Rodney J. Baxter — Research Achievements

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Baxter 2025: Exactly solved models and beyond . . .

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Outline

Baxter's work has involved solving highly non-trivial lattice models in the most brilliant way.

I will briefly review the works on

- Yang-Baxter equation and exact solutions
- Eight-vertex model
- Restricted solid-on-solid models (RSOS)
- Chiral Potts model
- 3D lattice models

From Baxter's book:

"I suppose the justification for studying these lattice models is very simple: *they are relevant and they can be solved, so why not do so and see what they tell us?*"

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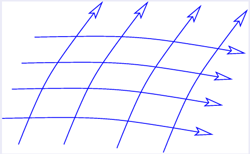
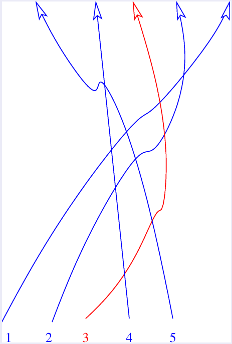
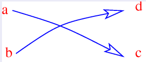
Introduction to Yang-Baxter equation

- oriented 4-valent graph
- edges carry "spin" variables $a, b, c, \ldots = \{1, 2, \ldots, N\}$





- local Boltzmann weights



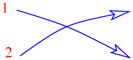
$= R_{ab}^{cd}$

- Partition function

$$Z = \sum_{(spins)} \prod_{(vertices)} R_{ab}^{cd}$$

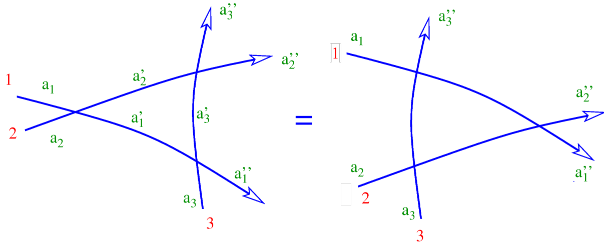
- sum over internal spins (ext. spins fixed)

- R_{ab}^{cd} are different for different vertices



$= R_{12}\,, \qquad R_{12} = R(u_1 - u_2)$

Yang-Baxter equation (=equality of partition functions of two triangles)



$$\sum_{a_1',a_2',a_3'} (R_{12})^{a_1',a_2'}_{a_1,a_2} (R_{13})^{a_1'',a_3'}_{a_1',a_3} (R_{23})^{a_2'',a_3''}_{a_2',a_3'} = \sum \ldots$$

Operator form

$$R_{12}\ R_{13}\ R_{23} = R_{23}\ R_{13}\ R_{12}, \qquad R_{ij} = R(u_i - u_j)$$

$$R_{12}, R_{13}, R_{23}: \qquad V \otimes V \otimes V \to V \otimes V \otimes V$$

YBE leads to commuting transfer matrices and to Baxter's "Z-invariance" for arbitrary planar lattices

Exact solution of the 8-vertex model (Baxter 1971)

$$a : b : c : d = \mathrm{sn}(\mathrm{v} + \eta) : \mathrm{sn}(\mathrm{v} - \eta) : \mathrm{sn}(2\eta) : \mathrm{k}\,\mathrm{sn}(2\eta)\,\mathrm{sn}(\mathrm{v} + \eta)\,\mathrm{sn}(\mathrm{v} - \eta)$$

Parameters: v, η, k. (sn – elliptic sine function of modulus k)

Baxter's exact result for the free energy per site:

$$-\beta f \sim |T - T_c|^{2-\alpha} ,$$

$$\sim |T - T_c|^{2-\alpha} \, log|T - T_c| , \quad \text{if } \alpha/2 = \text{integer} .$$

$$T - T_c \sim k^2 \to 0 \qquad 2 - \alpha = \pi/2\eta$$

The "universality hypothesis" (Fisher 1966, Kadanoff 1969) states that

macroscopic properties of systems near a phase transition are independent of microscopic details of the interaction, depending only on global symmetries and dimensionality.

Baxter's exact result demonstrates a violation of the "universality" hypothesis.

The 8-vertex model inspirations for further developments

- Exact solution of the 8-vertex model (Baxter 1971)
  - Commuting transfer matrices
  - Functional equations and new algebraic structures
  - 8-vertex model is a "completely integrable quantum system" with an infinite number of commuting integrals of motion.
- Quantum Inverse Problem Method for solving models of statistical mechanics and quantum field theory (Faddeev-Sklyanin-Takhtajan'79)
- Connection to Classical Inverse Problem Method and theory of solitons. Classical Yang-Baxter eq. (Sklyanin'80)

$$R(u, \hbar) = 1 + \hbar r(u) + O(\hbar^2), \qquad \hbar \to 0$$

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

- Factorized S-matrices in 2D field theory and theory of "quantum solitons". (Zamolodchikov-Zamolodchikov'79)
- Connection to representation theory (Kulish-Reshetikhin-Sklyanin'81)

Solutions to the Yang-Baxter equation

YBE is an overdetermined system of algebraic equations (N^6 equations for $3\,N^4$ unknowns). We do not know a general solution even for $N = 2$!

- Known solutions (various methods):Onsager, McGuire, Yang, Baxter, Cherednik, Korepin, Izergin, Takhtajan, Perk, Schultz, Fateev, Zamolodchikov(s), Kulish, Reshetikhin, Kirillov, Sklyanin, Smirnov, Belavin, Drinfeld, McCoy, Au-Yang, Stroganov, Andrews, Forrester, Bazhanov, Jimbo, Kashiwara, Miwa, Date, Okado, Kuniba, Miki, Nakanishi, Hasegawa, Yamada, Pearce, Warnaar, Seaton, Nienhuis, Lukyanov, Faddeev, Volkov, Ge, Mangazeev, Kashaev, Akutsu, Deguchi, Shiroishi, Wadati, Sergeev, Khoroshkin, Stolin, Tolstoy, Teschner, Tsuboi, Derkachov, Zabrodin, Lukowski, Frassek, Meneghelli, Saleur, Staudacher, Shastry, Shadrikov, Kels, Pasquier, Hibberd, Fendley, Spiridonov, Yamazaki
- Statistics: Russian 40%, Japanise 20%, English 15%, German 5%, Dutch 5%, Chinese 5%, . . . , Norwegian 1.5%
- Original works:
  - 1964 J.B.McGuire "Study of Exactly Soluble One-Dimensional N-Body Problems"
  - 1967 C.N.Yang "One-dimensional N-body problem with δ-function interaction
  - 1972 R.J.Baxter "Partition function of the eight-vertex lattice model"

Faddeev-Takhtajan 1979:
"...therefore it is natural to call the formula $\mathbf{R}_{12}\mathbf{L}_2\mathbf{L}_1 = \mathbf{L}_1\mathbf{L}_2\mathbf{R}_{12}$ as the "Baxter-Yang relation"

P.Kulish: equations can be solved, but relations can only be verified.

Conformal invariance of critical fluctuations and Andrews-Baxter-Forrester lattice models

- It was proposed that statistical systems at critical points are not only scale invariant but also **conformally invariant**. (Polyakov, Kadanoff, Wilson 1969)
- In two dimensions the group of conformal transformations is infinite-dimensional. Using this fact Belavin-Polyakov-Zamolodchikov (1984) constructed an infinite set of "minimal" conformal field theories (CFT's) and explicitly calculated the spectrum of scaling dimensions for all local fields.
- About the same time (Baxter-Pearce, 1982) exactly solved "hard hexagons" model and then Andrews-Baxter-Forrester (1984) solved an infinite set of "Restricted Solid-on-Solid" (RSOS) lattice models and calculated their critical exponents.
- Huse (1984) has shown that the RSOS critical exponents exactly match the scaling dimensions in a subset of minimal CFT's selected by the unitarity requirement (Friedan, Qiu, Shenker, 1984).

G.E.Andrews, R.J.Baxter and P.J.Forrester, Palm Cove, Australia, 2015.

Do we really understand algebraic structure of the YBE?

- Universal R-matrix for quantized algebra $\mathcal{A}$ (Drinfeld'87)

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}, \qquad \mathcal{R} \in \mathcal{B}_-(\mathcal{A}) \otimes \mathcal{B}_+(\mathcal{A})$$

- Despite the existence of algorithmic recipies for universal R-matrix (e.g. (Khoroshkin-Tolstoy'92) formula for affine Lie algebras) it is mostly impossible to calculated it explicitly.

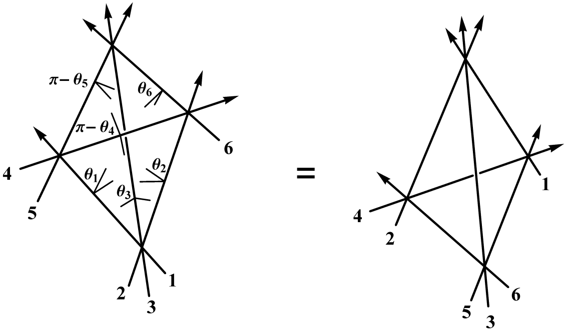
- Additional insights are needing to fully comprehend solutions of YBE.

- **A smart idea**: maybe they all come from extra dimensions???.

Integrable lattice systems in 3 dimensions

- Tetrahedron equation is a 3D generalisation of YBE

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$



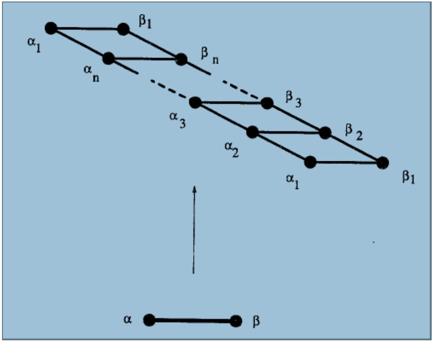
- First non-trivial solution obtained by Zamolodchikov in 1980 for "factorized S-matrix of scattering of stright strings in 2+1 dim.
- Method: An extraordinary feat of intuition (as described by Baxter)
- Partition function of this 3D model exactly calculated by Baxter 1984.

Chiral Potts model

- "Commuting transfer matrices in the chiral Potts models: Solutions of the star-triangle equations with genus > 1" (3-state spins) (Au-Yang-McCoy- Perk-Tang-Yan, 1987)
- Generalization to N-valued spins $\alpha \in Z_N$, for any $N \geq 2$ (Baxter-Perk-Au-Yang 1988)
- Calculation of the free energy (Baxter 1988)
- Generalized chiral Potts model with multiple spins at each lattice $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z_N \otimes Z_N \otimes \ldots \otimes Z_N$. (VB-Kashaev-Mangazeev-Stroganov 90'), (Date-Jimbo-Miki-Miwa 90')
- Generalized chiral Potts model has a 3D intepretation! (VB-Baxter'92).

A hidden third dimension

In the generalized chiral Potts model spin take N^n values. Introduce n spin variables at each site of 2D lattice $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in Z_N$.



- Edge weights of generalized chiral Potts "localize" in the 3rd dimension!
- New solvable N-state model in 3D, containing Zamolodchikov's model for $N = 2$ (VB-Baxter'92).
- Partition function calculated exactly (VB-Baxter'93).

Baxter's work on the Yang-Baxter equation has also led to

- the invention of quantum groups by Drinfeld and Jimbo, who have been honoured for their work by the Fields Medal for Drinfeld in 1990 and the Wigner Medal for Jimbo in 2010
- the discovery of a knot invariant by Jones, who was honoured by the Fields Medal in 1990
- the connection of Gauge/String theory and 2D integrable system (Maldacena, Minahan-Zarembo, Costello-Witten-Yamazaki).

Conclusion

Isaac Newton said “If I have seen further, it is by standing on the shoulders of giants.”. He was referring to Copernicus, Galileo and Kepler.

There is no doubt that Prof. Baxter is a giant who has brought the torch of mathematical physics into the 21-st century.

Rephrasing Newton, I would say, “We are able to see further, because of the outstanding work of Professor Baxter”.