Metrical properties of exponentially growing partial quotients.

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Joint work with Mumtaz Hussain

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The central result in the theory of Diophantine approximations is the Dirichlet's Theorem.

Theorem (Dirichlet, 1842)

For any $x \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN} \quad \text{and} \quad 1 \le q \le N.$$

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Corollary

For any irrational $x \in \mathbb{R}$ there infinitely many $p, q \in \mathbb{Z}$, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Continued fraction

Every irrational number $x \in (0, 1)$ has a unique infinite continued fraction expansion

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_1, a_2, ...]$$

Denote its nth convergent by

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It turns out that convergents provide explicit solutions to the Corollary of Dirichlet's Theorem; that is,

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Surprisingly, the opposite is also (almost) true, namely there is a following result.

Theorem

If for an irrational number $x \in \mathbb{R}$, we have that

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$$

for a rational number $\frac{p}{q}$, then $\frac{p}{q} = \frac{p_n}{q_n}$ for some n.

Can we improve the function $1/q^2$?

This leads us to considering the following set

$$\mathcal{K}(\Psi) = \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^2\Psi(q)} \quad \text{for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}$$

Theorem (Khintchine, 1924)

Lebesgue measure of the set $\mathcal{K}(\Psi)$ satisfies the following 0-1 law:

$$\lambda(\mathcal{K}(\Psi)) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} (q\Psi(q))^{-1} < \infty$$

$$\lambda(\mathcal{K}(\Psi)) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} (q\Psi(q))^{-1} = \infty$$

Notice that the Lebesgue measure is zero for $\mathcal{K}(\Psi) = q^{\tau}$ for any $\tau > 0$, and Khintchine's theorem gives no further information about the size of the set $\mathcal{K}(\Psi)$.

The growth rate of partial quotients plays main role in approximation properties due to a corollary of the Perron formula:

Lemma

For all irrational numbers $x = [a_1, \ldots, a_n, \ldots]$ *and for all convergents* $\frac{p_n}{q_n}$ *to* x*, we have*

$$\frac{1}{(a_{n+1}+2)q_n^2} < \frac{1}{q_n(q_{n+1}+q_n)} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1} q_n^2}$$

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Because of this, set $\mathcal{K}(\Psi)$ for $\Psi(q) = q^\tau$ can be rewritten as

$$\{x \in [0,1) : a_{n+1}(x) \geq \Psi(q_n) \quad \text{for infinitely many } n \in \mathbb{N}\}$$

Improving the Dirichlet

Alternatively, if one wants to improve an 'original' Dirichlet's theorem, one can consider a set of ψ-Dirichlet improvable numbers:

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\, N \text{ such that the system } |qx - p| < \psi(t),\ |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\}.$$

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Denote $\Phi(t) = \frac{t\psi(t)}{1 - t\psi(t)}$. Then for non-increasing function ψ with $t\psi(t) < 1$ we have

Lemma (Kleinbock-Wadleigh)

Let $x \in [0, 1) \setminus \mathbb{Q}$. Then

(i) $x \in D(\psi)$ *if* $a_{n+1}(x)a_n(x) \le \Phi(q_n)/4$ *for all sufficiently large n.*

(ii) $x \in D^c(\psi)$ *if* $a_{n+1}(x)a_n(x) > \Phi(q_n)$ *for infinitely many n.*

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This tell us that sometimes it is important to know the growth rate of the product of consecutive partial quotients.

$\Phi(q_n) \to \Phi(n)$

There exists $w > 1$ such that for every $x \notin \mathbb{Q}$, $w^n \leq q_n(x)$ for all $n \geq 2$. There also exists $W > w$ such that for almost every x, $q_n(x) \leq W^n$ for all large enough n.

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This fact naturally suggests to consider sets of the form

$$\mathcal{E}_1(\psi) := \{x \in [0,1) : a_n(x) \geq \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

or

$$\mathcal{E}_2(\psi) := \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \psi(n) \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

and many others.
We want to analyse them in terms of Lebesgue measure and Hausdorff dimension.

Borel-Bernstein theorem

Theorem (Borel-Berstein, 1912)

The Lebesgue measure of the set

$$\mathcal{E}_1(\psi) := \{x \in [0,1) : a_n(x) \geq \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

is either zero or full depending upon the convergence or divergence of the series $\sum_{n=1}^{\infty} \psi(n)^{-1}$ *respectively.*

Preparations for formulating some known results.

It turns out that the most important case is when $\psi(n) = B^n$ for $B > 1$. Consider an equation

$$\sum_{1 \le a_1, \ldots, a_n \le M} \frac{1}{q_n^{2x}} = B^{nx}. \tag{1}$$

It has a unique solution $x = s_{n,M,B}$. It is possible to show that the limit of $s_{n,M,B}$ when $n, M \to \infty$ exists. Denote it by $s(B)$.

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The Hausdorff dimension of this set is given by

Theorem (Wang-Wu, 2008)

For any $1 \le B < \infty$,

$$\dim_{\mathrm{H}} \mathcal{E}_1(B) = s(B).$$

For different applications we will change the function B^{nx} on the right hand side of (1) to other functions.

Some sets to consider 1

Consider sets

$$\mathcal{E}_2 = \{x \in [0,1) : a_n(x)a_{n+1}(x) \geq B^n \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

and $\mathcal{F}(B) := \mathcal{E}_2(B) \setminus \mathcal{E}_1(B)$, so that

$$\mathcal{F}(B) = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < B^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

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It is known that $\mathcal{F}(B)$ has positive Hausdorff dimension. More precisely, one has

Theorem (Bakhtawar-Bos-Hussain, 2020)

$$\dim_{\mathrm{H}} \mathcal{F}(B) = \dim_{\mathrm{H}} \mathcal{E}_2 = t_B, \tag{2}$$

where the corresponding r.h.s. of (1) *is equal to* B^{nx^2}.

Some sets to consider 2

Previous result was recently generalised. Consider the set $\mathcal{F}(B_1, B_2) := \mathcal{E}_2(B_1) \setminus \mathcal{E}_1(B_2)$, so

$$\mathcal{F}(B_1, B_2) = \left\{ x \in [0,1) : \begin{array}{r} a_{n+1}(x)a_n(x) \geq B_1^n \text{ for infinitely many } n \in \mathbb{N} \text{ and} \\ a_{n+1}(x) < B_2^n \text{ for all sufficiently large } n \in \mathbb{N} \end{array} \right\}.$$

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Theorem (Hussain-Li-Sh., 2023)

For any $B_1, B_2 > 1$, we have

- *if $B_1^{t_{B_1}} \leq B_2$, then $\dim_H \mathcal{F}_{B_1,B_2} = t_{B_1}$;*
- *if $B_1^{t_{B_1}} \geq B_2 > B_1^{1/2}$, then $\dim_H \mathcal{F}_{B_1,B_2} = g_{B_1,B_2}$;*
- *if $B_1^{1/2} \geq B_2$, then $\mathcal{F}_{B_1,B_2} = \emptyset$,*

where for g_{B_1,B_2} the corresponding r.h.s. of (1) is equal to $\frac{B_1^{nx}}{B_2^{(1-x)n}}$.

Some sets to consider 3

Another example is the main result from paper by Huang-Wu-Xu. They have considered a set

$$E_m(B) := \{x \in [0,1) : a_n(x)a_{n+1}(x)\cdots a_{n+m-1}(x) \geq B^n \text{ for infinitely many } n \in \mathbb{N}\}.$$

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At the heart of their paper is the following result.

Theorem (Huang-Wu-Xu, 2020)

For $1 \leq B < \infty$, and any integer $m \geq 1$,

$$\dim_{\mathrm{H}} E_m(B) = t_B^{(m)}, \tag{3}$$

where the corresponding r.h.s. of (1) is equal to $B^{nf_m(x)}$ and $f_m(s)$ is given by the following iterative formula:

$$f_1(s) = s, \quad f_{k+1}(s) = \frac{sf_k(s)}{1 - s + f_k(s)}, \ k \geq 1.$$

How are those results proved?

Hausdorff dimension is usually found in two steps: the upper bound and the lower bound.
The lower bound can be found from the Mass Distribution principle formulated below.

Lemma

Let $E \subset [0,1)$ be a Borel set and μ be a measure with $\mu(E) > 0$, suppose that for some $s > 0$, there is a constant $c > 0$ such that for any $x \in [0,1)$ one has

$$\mu(B(x,r)) \leq cr^s, \tag{4}$$

where $B(x,r)$ denotes an open ball centred at x and radius r, then $\dim_{\mathcal{H}} E \geq s$.

Main result

For a fixed integer number m and for all integers $0 \leq i \leq m-1$, let $A_i > 1$ be a real number. Define the set

$$S_m(A_0, \ldots, A_{m-1}) = \{x : \ c_i A_i^n \leq a_{n+i}(x) < 2c_i A_i^n, 0 \leq i \leq m-1, \text{ for i. m. } n \in \mathbb{N}\}$$

where $c_i > 0 \in \mathbb{R}$. For any $0 \leq i \leq m-1$, define the quantities

$$\beta_{-1} = 1, \ \ \beta_i = A_0 \cdots A_i.$$

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Define d_i as a limit for $n, M \to \infty$ of solution of the equation

$$\sum_{1 \leq a_1, \ldots, a_n \leq M} \frac{1}{q_n^{2x}} = \frac{\beta_i^{nx}}{\beta_{i-1}^{n(1-x)}}. \tag{5}$$

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Theorem (Hussain-Sh., 2023)

$$\dim_{\mathrm{H}} S_m = \min_{0 \le i \le m-1} d_i$$

Why is this set interesting?

The set $S_m(A_0, \ldots, A_{m-1})$ for a suitable choice of the parameters m, A_i, c_i is a subset of all of the previously listed sets and has the same Hausdroff dimension as them.
For example, consider a set

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To get a lower bound in this setup using our result, we set $m = 1, A_0 = B$, that is we consider the set

$$S_1(B) = \{x \in [0, 1) : B^n \leq a_n(x) \leq 2B^n \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

which is clearly a subset of $\mathcal{E}_1$.

Thank you!