Statistics of matrix elements in integrable models

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F Essler and A deKlerk PRX '24

R. Senese and F Essler '25

Baxter2025: Exactly Solved Models and Beyond

EPSRC Engineering and Physical Sciences Research Council

Cairns, 2010.

“Generic” many-particle systems (with local Hamiltonians)

Relaxation of observables to equilibrium

↕

Structure of local operators in the basis of energy eigenstates

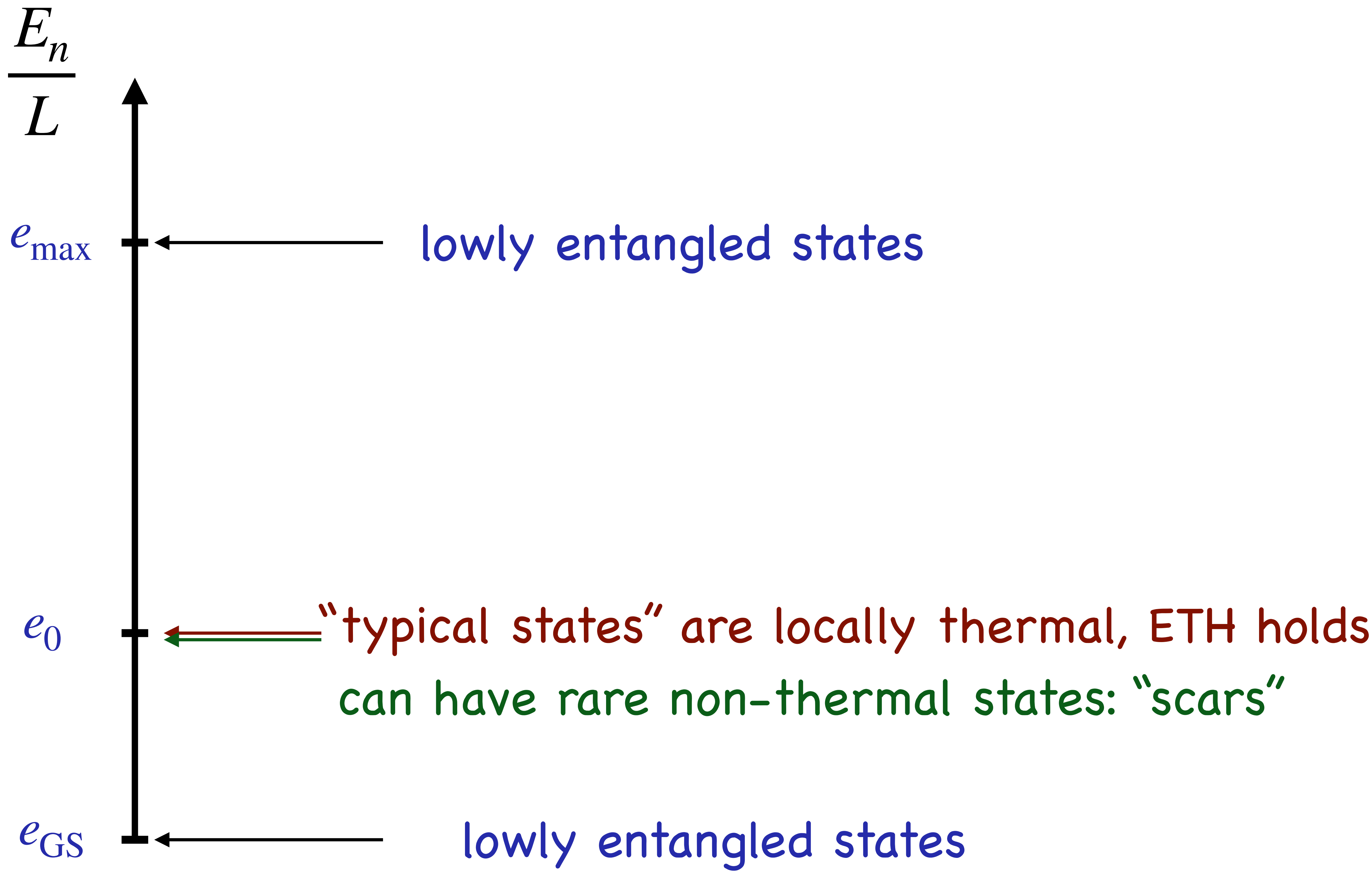
$H|E_n\rangle = E_n|E_n\rangle$, H has local density, $\mathcal{O}$ local operator

$\mathcal{O}_{n,m} = \langle E_n|\mathcal{O}|E_m\rangle$

PDF$\left(\mathcal{O}_{n,m}\right)$ fulfils **Eigenstate Thermalisation Hypothesis**

Structure of energy-eigenstates in "generic" models

e.g. lattice model of spins with "local" Hamiltonians



Part I: What is PDF$\left(\langle E_n | \mathcal{O} | E_m \rangle\right)$ in integrable models ?

While we believe our findings are more general, they are obtained for the repulsive Lieb-Liniger model:

$$H(c) = \int \mathrm{d}x \Big(-\Phi^\dagger(x)\partial_x^2\Phi(x) + c\big(\Phi^\dagger(x)\big)^2\big(\Phi(x)\big)^2 \Big)$$

$\Phi(x)$ canonical Bose field **c>0**

Reason to work with LL is technical: no bound states ("strings")

Integrable models: conservation laws

Extensive # of mutually compatible "local" conservation laws

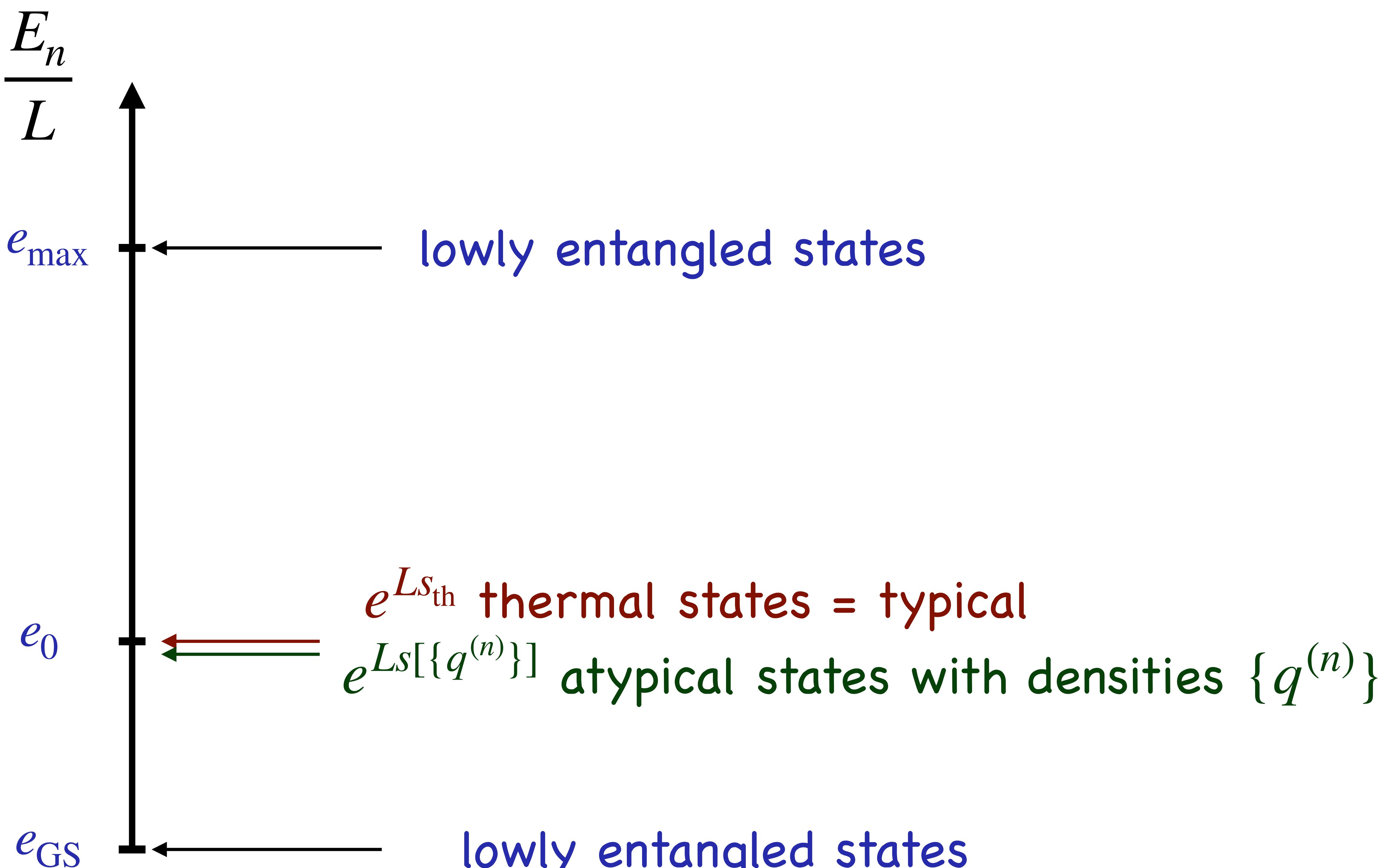
$$Q^{(n)} = \sum_j Q_j^{(n)} \;, 1 \le n \le N \qquad [Q^{(n)}, H] = 0 = [Q^{(n)}, Q^{(m)}]$$

↑

spatially local

Structure of energy-eigenstates in integrable models

e.g. lattice model of spins with "local" Hamiltonians



Micro-states

Lieb&Liniger '63

Bethe Ansatz gives simultaneous eigenstates of all cons. laws:

$$H|\lambda_1,\ldots,\lambda_N\rangle = \underbrace{\left(\sum_{k=1}^{N}\lambda_j^2 - \mu\right)}_{E(\lambda)}|\lambda_1,\ldots,\lambda_N\rangle$$

Bethe equations

$$\lambda_j L + \sum_{n=1}^{N} 2\arctan\left(\frac{\lambda_j - \lambda_n}{c}\right) = 2\pi I_j$$

j=1,...,N; I_j (half-odd) integers

$$\{I_j\} \longleftrightarrow \{\lambda_j\} \longleftrightarrow |\lambda_1,\ldots,\lambda_N\rangle$$

I_j are "quantum numbers" that characterise energy eigenstates

Matrix elements are known

Slavnov '89

$$\rho(x) = \Phi^\dagger(x)\Phi(x)$$

$$|\langle \lambda_1, \ldots \lambda_N | \rho(0,0) | \mu_1, \ldots, \mu_N \rangle|^2 =$$

$$\frac{\left(\sum_{i=1}^N \mu_i - \lambda_i\right)^2}{L^{2N}\mathcal{N}_{\boldsymbol{\lambda}}\mathcal{N}_{\boldsymbol{\mu}}} \frac{\prod_{i\neq j}(\lambda_i - \lambda_j)(\mu_i - \mu_j)}{\prod_{i,j}(\lambda_i - \mu_j)^2} \prod_{i\neq j} \frac{\lambda_i - \lambda_j + ic}{\mu_i - \mu_j + ic}$$

$$\times \left| \det_{i,j\neq p} \left[(V_i^+ - V_i^-)\delta_{ij} + i(\mu_i - \lambda_i) \prod_{k\neq i} \frac{\mu_k - \lambda_i}{\lambda_k - \lambda_i} \left(\frac{2c}{(\lambda_i - \lambda_j)^2 + c^2} - \frac{2c}{(\lambda_p - \lambda_j)^2 + c^2} \right) \right] \right|^2$$

where

$$V_i^\pm = \prod_{k=1}^N \frac{\mu_k - \lambda_i \pm ic}{\lambda_k - \lambda_i \pm ic}$$

$$\mathcal{N}_{\boldsymbol{\lambda}} = \det_{i,j=1,\ldots,N} \left[\delta_{ij} \left(1 + \frac{1}{L} \sum_{k=1}^N \frac{2c}{c^2 + (\lambda_i - \lambda_k)^2} \right) - \frac{1}{L} \frac{2c}{c^2 + (\lambda_i - \lambda_j)^2} \right]$$

Organising principle: Macro-states

Macro-states = families of micro-states in large L that have the same densities $\{q^{(n)}\}$ of all conservation laws up to finite-size corrections

Characterised by distribution function $\varrho(z) > 0$:

$$L\varrho(z)\Delta z = \text{ number of } \frac{I_j}{L} \text{ in } [z, z + \Delta z]$$

Computation of PDF$\left(\langle E_n | \mathcal{O} | E_m \rangle\right)$:

1. Choose a local operator.

2. Fix two macro-states $\varrho_{1,2}(z)$

3. Sample corresponding micro-states (fulfil Bethe eqns):

   Generate sets $\{J_j | j = 1, \ldots N\}$ of (half-odd) integers that correspond to $\varrho_{1,2}(z)$ and solve Bethe equations for $\{\lambda_j\},\{\mu_j\}$

4. Use exact expressions for matrix elements to compute them

5. Repeat steps 3 &4 to obtain statistics

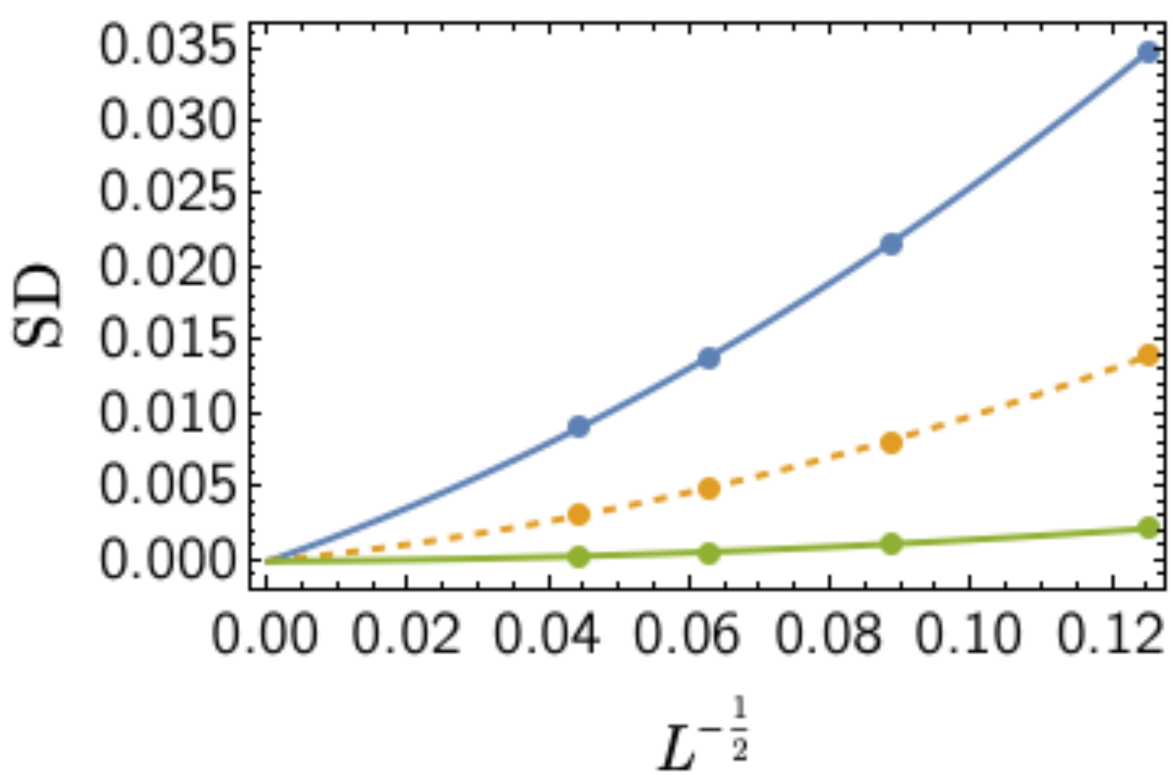
(correctly sampling macro-states is non-trivial)

Results: Diagonal matrix elements

$$\langle \boldsymbol{J} | \mathcal{O} | \boldsymbol{J} \rangle = f_{\mathcal{O}}[\rho] + o(L)$$

Depend only on macro-state, i.e. $\varrho(z)$, in thermodyn. limit.

cf Korepin et al 80ies



$\langle \boldsymbol{J} | (\phi^{\dagger}(0))^2 (\phi(0))^2 | \boldsymbol{J} \rangle$
for T=10 and c=1,4,16

cf Alba '15

Results II: Off-diagonal matrix elements

- $|I\rangle, |J\rangle$ belonging to **different** macro-states $\varrho_{0,1}$

$$|\langle I|\mathcal{O}(0)|J\rangle|^2 \propto e^{-c^{\mathcal{O}}_{\varrho_0,\varrho_1}L^2}$$

for all ME

- $|I\rangle, |J\rangle$ belonging to the same (thermal) macro-state

$$|\langle I|\mathcal{O}(0)|J\rangle|^2 \propto e^{-c_0 L\ln(L) - M_{\lambda,\mu}L}$$

for **typical** MEs

interesting PDF for $M_{\lambda,\mu}$

$$|\langle I|\mathcal{O}(0)|J\rangle|^2 \propto L^{-c_1}e^{-c_2 L}$$

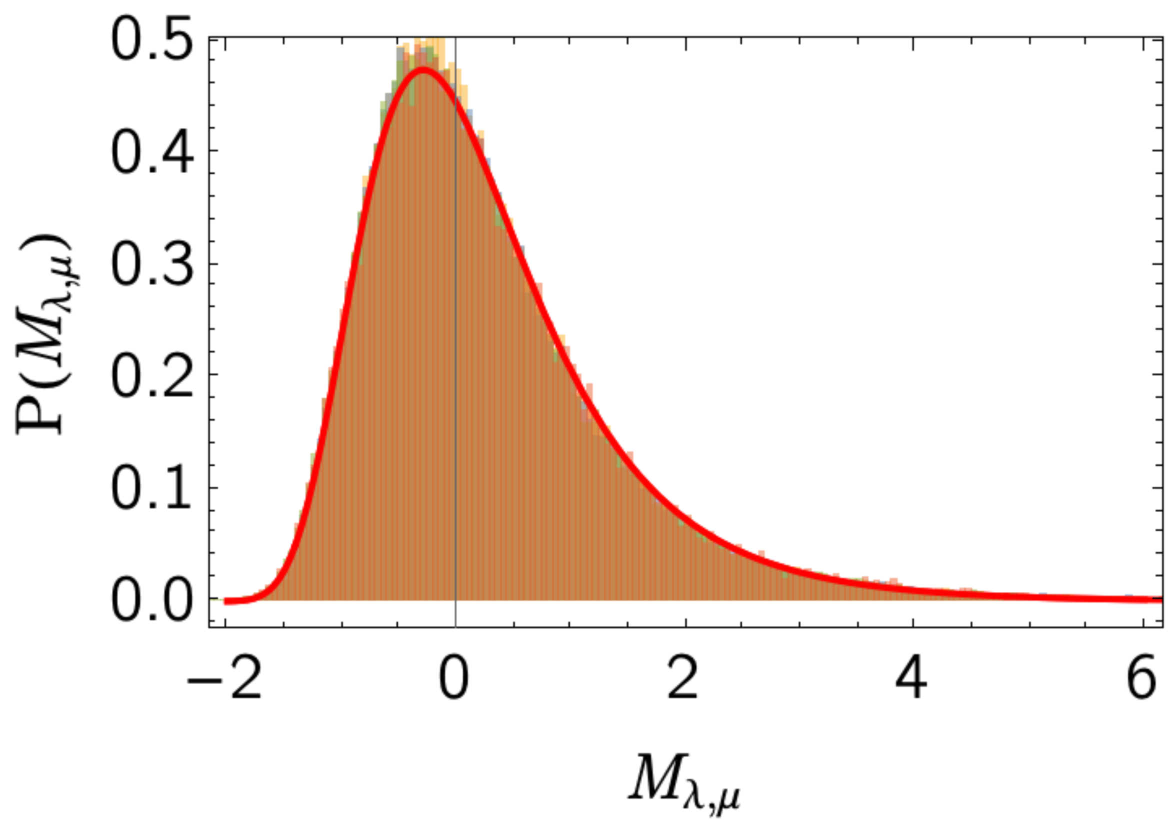
for **rare** MEs

- $|I\rangle, |J\rangle$ belonging to **same** macro-state

$$|\langle I|\mathcal{O}(0)|J\rangle|^2 \propto e^{-c_0 L \ln(L) - M_{\lambda,\mu} L}$$

for **typical** MEs

$\Phi(0)$ at $c = \infty$ and thermal state with T=10, D=1:

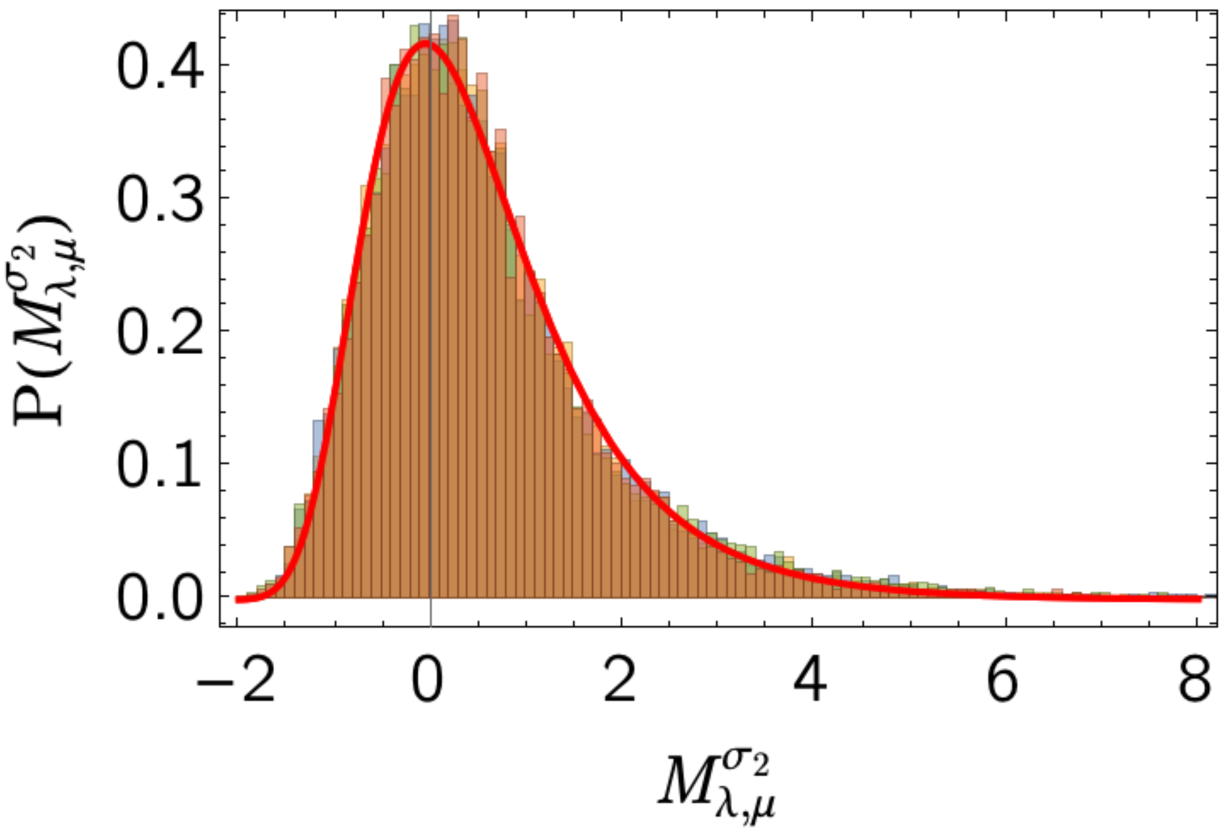


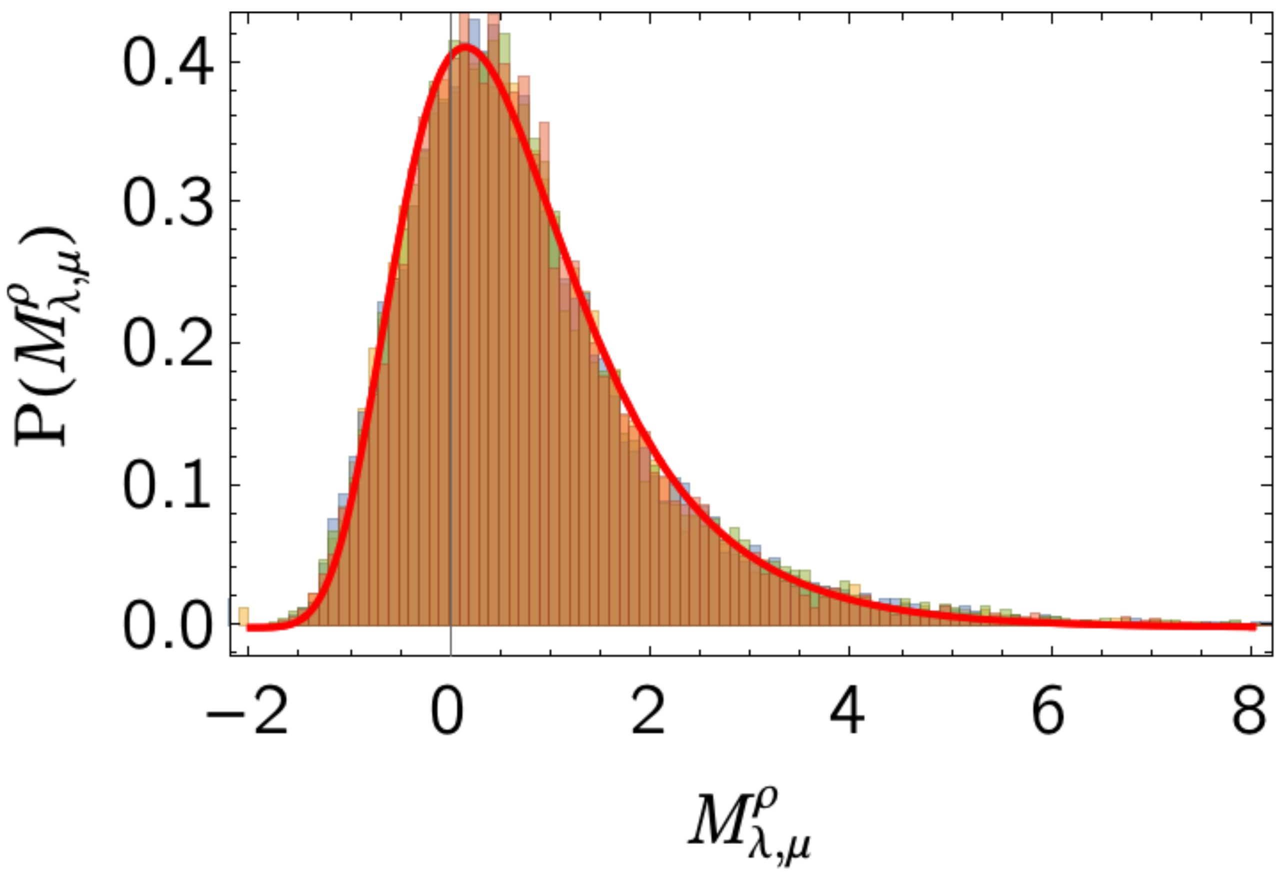
scaling collapse for L=64,128,256,512

Fit is to Frechet distribution

$$P_{\alpha,\beta,\nu}(x) = \begin{cases} (x-\nu)^{-\alpha-1} \exp\left[-\left(\frac{x-\nu}{\beta}\right)^{-\alpha}\right] & \text{if } x > \nu \\ 0 & \text{else.} \end{cases}$$

$\left(\Phi^\dagger(0)\right)^2\left(\Phi(0)\right)^2$ and $\Phi^\dagger(0)\Phi(0)$ at $c = 4$ and thermal state with T=5, D=1:





scaling collapse for L=64,96,160,224

Fits to Frechet distributions

Upshot: **typical** MEs not relevant for local correlation fns.

$$\langle \boldsymbol{J} \,|\, \Phi^{\dagger}(x,t)\Phi(0,0) \,|\, \boldsymbol{J} \rangle = \sum_{\boldsymbol{I}} |\langle \boldsymbol{I} \,|\, \Phi(0,0) \,|\, \boldsymbol{J} \rangle|^2 e^{it(E_{\boldsymbol{J}}-E_{\boldsymbol{I}})-ix(P_{\boldsymbol{J}}-P_{\boldsymbol{I}})}$$

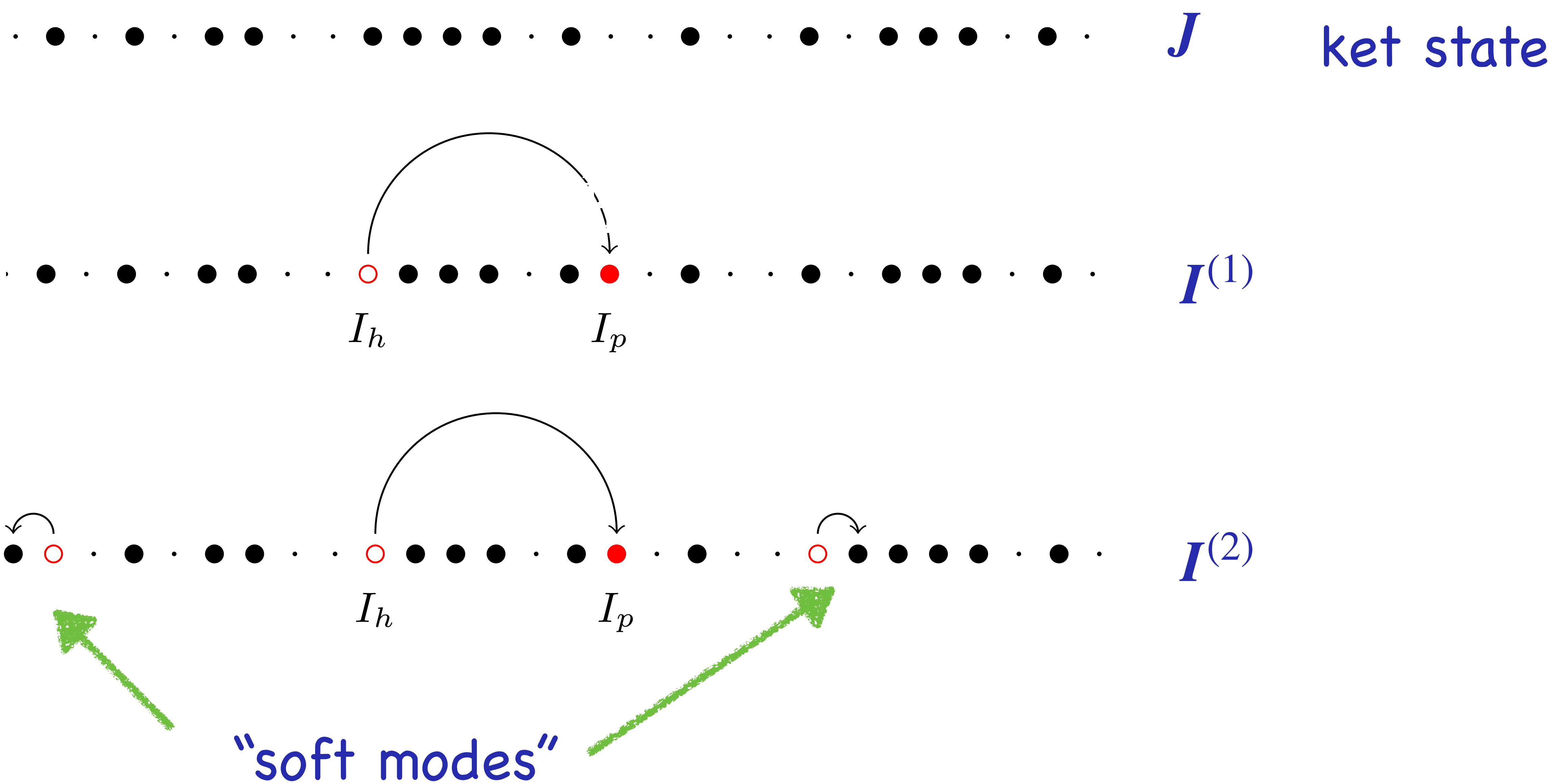
$\sim e^{Ls_T}$ states

$\sim e^{-c_0 L \ln(L) - M_{\lambda,\mu} L}$

Which states do contribute?

Rare large MEs and "soft modes"

Example: $\langle \boldsymbol{I} \,|\, \Phi^{\dagger}(x)\Phi(x) \,|\, \boldsymbol{J} \rangle$

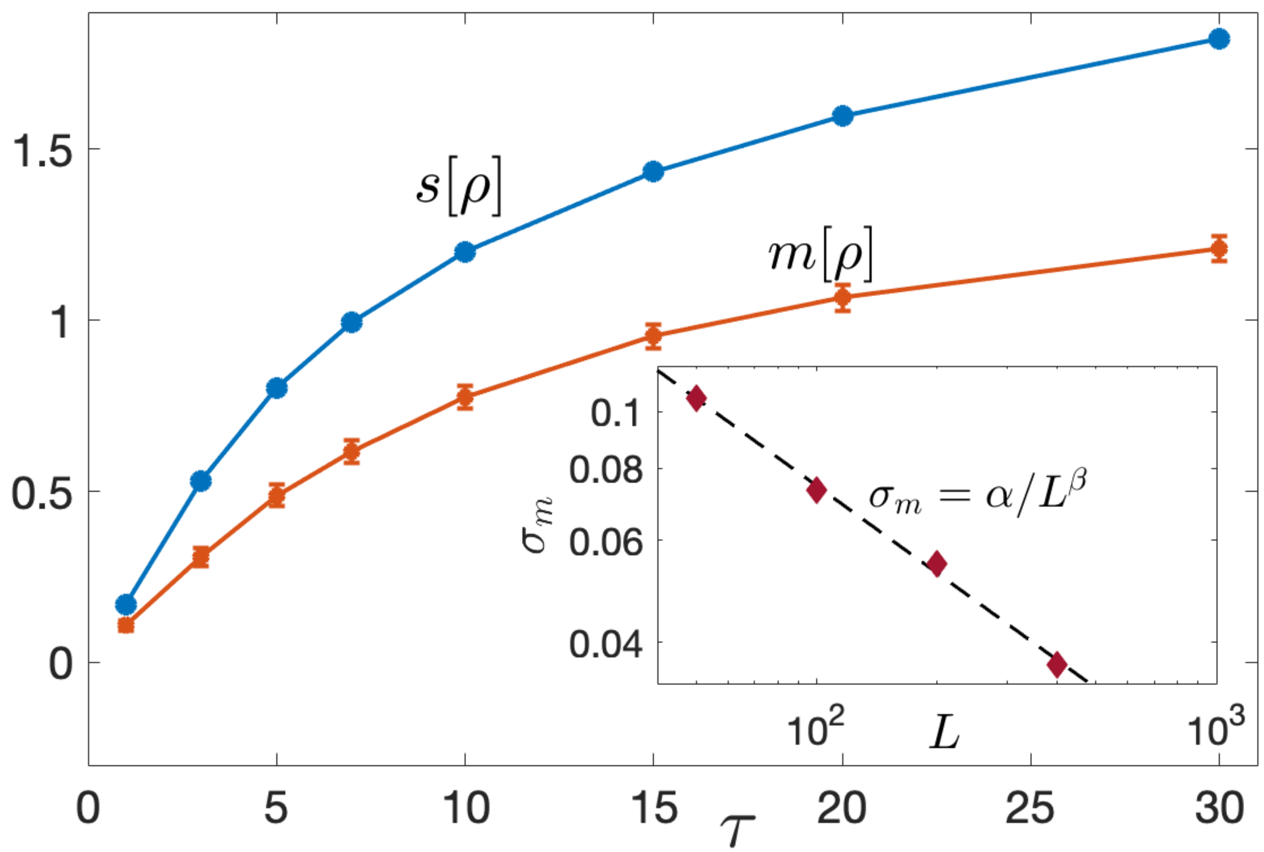


Metropolis Monte-Carlo algorithm

$$\langle \boldsymbol{J} \,|\, \Phi^{\dagger}(x,t)\Phi(0,0) \,|\, \boldsymbol{J} \rangle = \sum_{\boldsymbol{I}} |\langle \boldsymbol{I} \,|\, \Phi(0,0) \,|\, \boldsymbol{J} \rangle|^2 e^{it\left(E(\boldsymbol{J})-E(\boldsymbol{I})\right)-ix\left(P(\boldsymbol{J})-P(\boldsymbol{I})\right)}$$

$|\boldsymbol{J}\rangle$ a thermal microstate

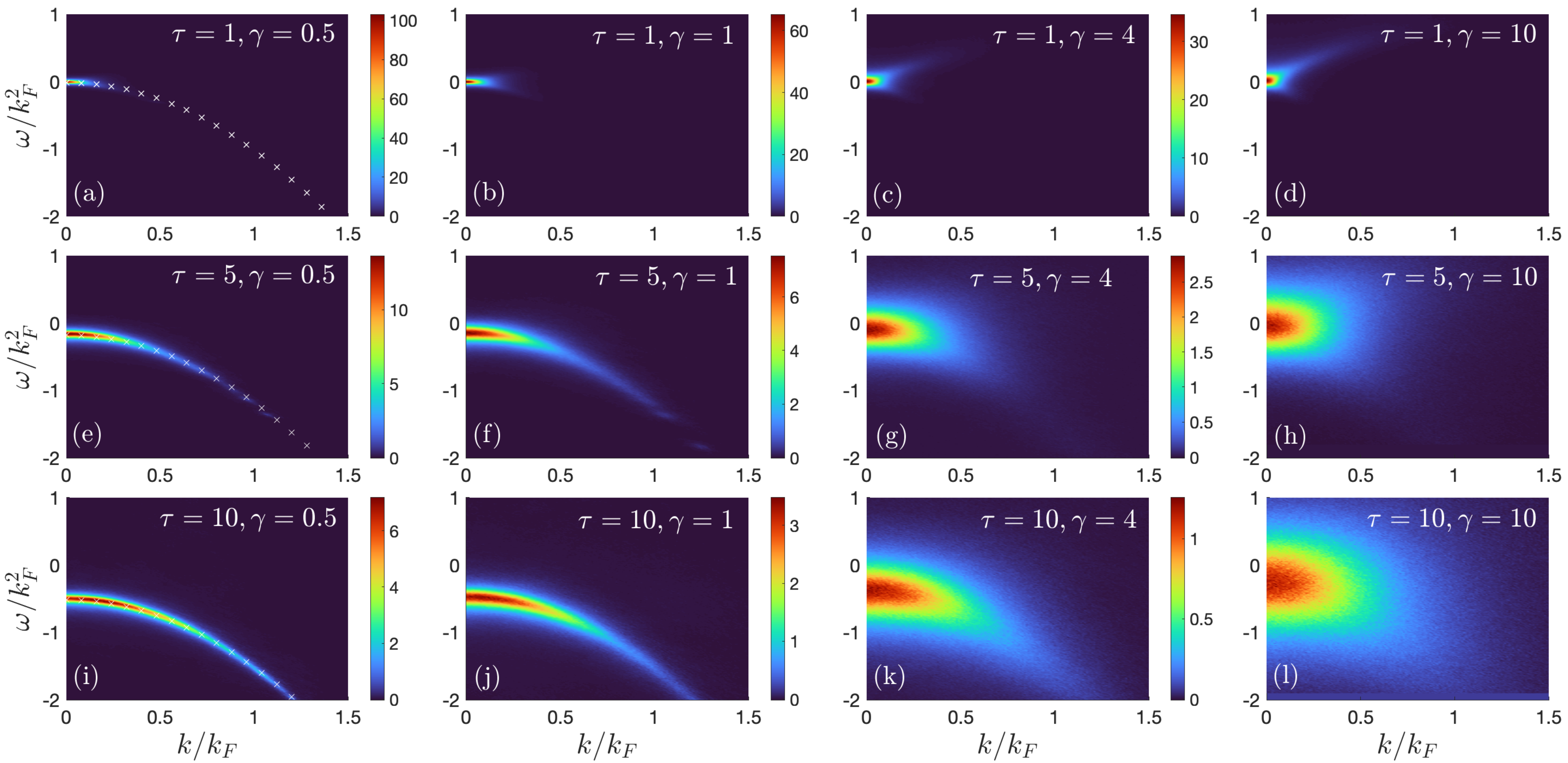
Sample $\boldsymbol{I}$ using Monte Carlo single-integer Metropolis update



exponentially many states contribute, but **sub-entropic**!

$\tau = T/n^2$

Results for L=N=200:



Benchmark at $c = \infty$ (Fredholm determinant) agrees extremely well.

Summary

- Statistics of matrix elements in energy eigenstates of integrable models has a rich and interesting structure
- Unlike in generic models the dynamics of local observables is governed by (exponentially many) **rare states.**
- Analytic results?
- Can sample rare states by Metropolis algorithm to get finite temperature dynamical 2-point function of Bose field.
- Generalizations to quantum quench, N-point functions? (is there a "sign-problem"?)