Time discretization of classical integrable models

Vincent Pasquier

IPhT, Saclay

Baxter 2025 Exactly Solved Models and Beyond: Celebrating the Life and Achievements of Rodney James Baxter

collaborators

- Enej Ilievski
- Tomaz Prozen
- Ziga Krajnik
- Benjamin Hery
- Gregoire Misguich
- Johannes Schmidt

Camberra 1989

Plan

- Motivation

- I – Discrete classical integrable systems
- II – Boxbball model
- III – Toda, Landau Lifshitz and critical models breaking Lorentz invariance

I - Motivation

- Transport out of equilibrium in quantum and classical systems

I - Motivation

- Transport out of equilibrium in quantum and classical systems
- time correlations in integrable models

I - Motivation

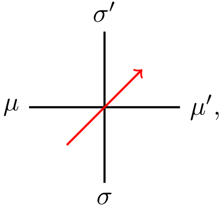
- Transport out of equilibrium in quantum and classical systems
- time correlations in integrable models
- intégrable $\Rightarrow$ analytic results? difference with nonintegrable?

interesting quantity:

Spin correlations.

Charge transport.

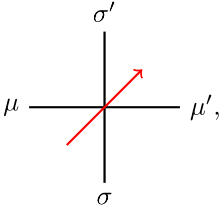
- The vertex:

$l_{\mu,\mu'}(u)=$ 

- Baxter considered positive Boltzman weights, the matrix $l_{\mu\sigma,\mu'\sigma'}$ (in the arrow direction) be unitary or stochastic.

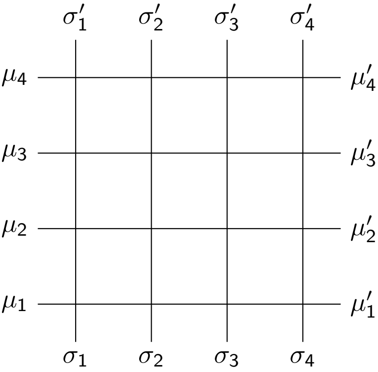
- The vertex:

$l_{\mu,\mu'}(u) =$



- Baxter considered positive Boltzman weights, the matrix $l_{\mu\sigma,\mu'\sigma'}$ (in the arrow direction) be unitary or stochastic.

partition function



Baxter transfer Matrix

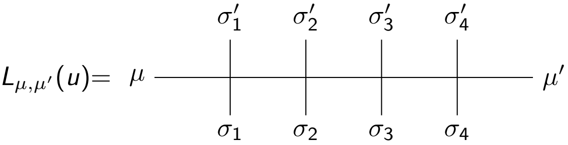


Figure: Baxter monodromy

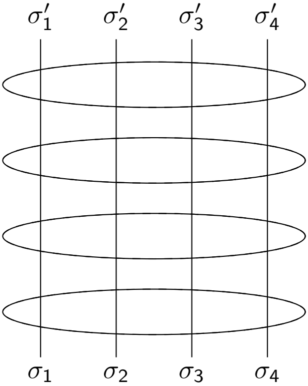
eight-vertex model: $\sigma = \pm 1$ physical space, $\mu = \pm 1$ auxiliary space.

- Closed transfer Matrix:

$$T(u) = \sum_m L_{\mu\mu}(u)$$

Rodney Baxter Used the Yang-Baxter equation to define commuting transfer matrices as a function of the spectral parameter.

cylinder partition function



- Stochasticity can be preserved by the transfer matrix but not unitarity (as far as I know).
- Nevertheless, Hamiltonian can be obtained and $T(u)$ can be viewed as the generating function of conserved qantities of a real time evolution, analogous to a Lax matrix.

Transfer map: Skyanin, Drinfeld

- Instead of a matrix, the vertex represents a map

$l_{\mu,\mu'}(u) =$

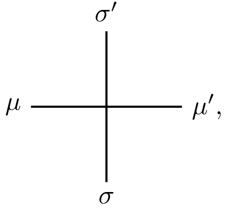


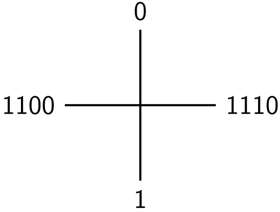
Figure: vertex

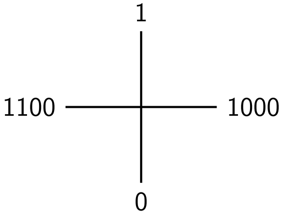
$$\mu, \sigma \to \mu', \sigma'$$

- The vertices cal be combined into a transfer propagator. Cyclicity requires $\mu = \mu'$ at the two ends of the chain

Discrete time evolution boxball: Takahashi and Satsuma

- Time evolution upwards analogous to a cellular automata:





- The carrier has n balls with $l \geq n \geq 0$, where l is its capacity. It scans the configuration of balls from left to right and picks up a ball if it can when there is one, leaving a ball when there is none.

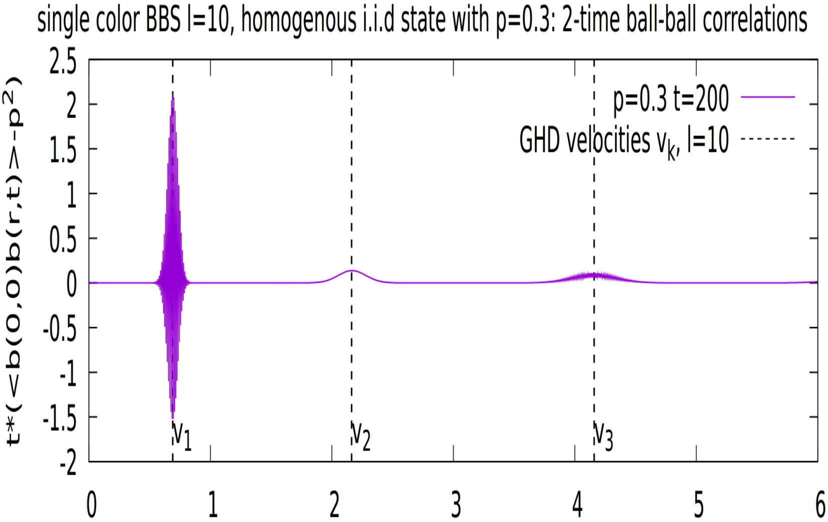
Example of Discrete time evolution boxball

- Cyclicity. One must make sure that after the last step, the load of the carrier coincides with its initial load.
- The model is integrable:
  Solitons made of k consecutive balls are conserved in number.
  propagators with different capacity (l) commute.
- Solitons travel ballistically with an effective speed v_k which can be exactly evaluated (GHD). The only necessary ingredients are the bare speed $v_j^0 = min(j, l)$, and the scattering length of two solitons of size j and k:

  $$\Delta_{k,l} = 2min(k, l)$$

  Time correlations can be evaluated.

correlation, Kuniba, Misguich, V.P.



Toda chain

- Toda Hamiltonian:

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2} + e^{q_{k+1} - q_k}$$

- $$\frac{d^2 q_j}{dt^2} = e^{q_{j+1} - q_j} - e^{q_j - q_{j-1}}$$

- Integrability results from a lax matrix constructed from a vertex matrix:

$$l_i(u) = \begin{pmatrix} u + p_i & -e^{q_i} \\ e^{-q_i} & 0 \end{pmatrix}$$

set $p_i = X_i$, $e^{q_i} = x_i$

Discrete time evolution Toda chain, Suris

- The vertex map , time goes up:

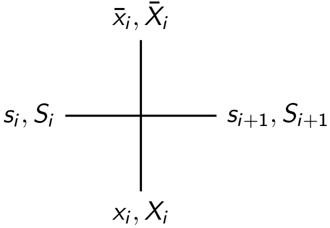


Figure: Time evolution

- The carrier has DST(discrete self trapping) variables S, s. It passes through the Toda configurations and updates the Toda variables x_i, X_i.
- Cyclicity. One must make sure that after the last step, $S_{N+1} = S_1$ and $s_{N+1} = s_1$

Discrete time evolution Toda chain

- How to define the vertex map?

Discrete time evolution Toda chain

- How to define the vertex map?
- use lax matrices: for Toda

$$l(u) = \begin{pmatrix} u + X & -x \\ \frac{1}{x} & 0 \end{pmatrix}$$

Discrete time evolution Toda chain

- How to define the vertex map?
- use lax matrices: for Toda

$$l(u) = \begin{pmatrix} u + X & -x \\ \frac{1}{x} & 0 \end{pmatrix}$$

- for DST

$$r(u) = \begin{pmatrix} u + Ss & -s \\ S & -1 \end{pmatrix}$$

It is necessary that both $l(u)$ $r(u)$ have the same Poisson bracket (Skyanin algebra).

Discrete time evolution Toda chain

- How to define the vertex map?
- use lax matrices: for Toda

$$l(u) = \begin{pmatrix} u + X & -x \\ \frac{1}{x} & 0 \end{pmatrix}$$

- for DST

$$r(u) = \begin{pmatrix} u + Ss & -s \\ S & -1 \end{pmatrix}$$

It is necessary that both $l(u)$ $r(u)$ have the same Poisson bracket (Skyanin algebra).

- Solve for Darboux transform:

$$l_i(u) r_i(u - \tau) = r_{i+1}(u - \tau) \bar{l}_i(u)$$

Discrete time evolution Toda

- Obtain the solution (Suris, Sklyanin):

$$X_j = -\tau + \frac{x_j}{\bar{x}_j} + \frac{s_{j+1}}{x_j}$$

$$\bar{X}_j = -\tau + \frac{x_j}{\bar{x}_j} + \bar{x}_j S_j$$

$$s_j = \bar{x}_j, \; S_{j+1} = \frac{1}{x_j}$$

- τ is the time step. In the τ to zero limit we recover the Toda equations of motion.
- Global map preserves the conserved quantities and is symplectic. To show canonicity construct generating function of canonical transform classical analogue of Baxter Q matrix.

$$F(x_i, \bar{x}_i, \tau) = -\tau \sum_i (\ln x_i - \ln \bar{x}_i) + \frac{x_i}{\bar{x}_i} - \frac{\bar{x}_{i+1}}{x_i}$$

Landau Lifshitz

- $$\frac{dS}{dt} = S \times \frac{d^2 S}{ds^2}$$

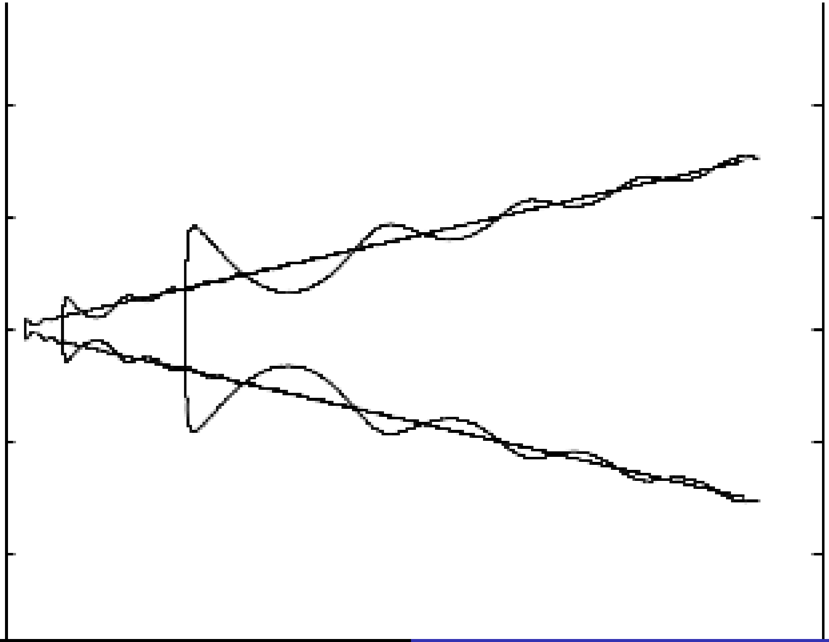
  where $S(s)$ is a unit vector $S_1^2 + S_2^2 + S_3^2 = 1$.
- Classical equation of motion for XXX spin chain.

Landau Lifshitz

- $$\frac{dS}{dt} = S \times \frac{d^2 S}{ds^2}$$
  where $S(s)$ is a unit vector $S_1^2 + S_2^2 + S_3^2 = 1$.
- Classical equation of motion for XXX spin chain.
- Local induction approximation for a filament in a superfluid. The filament is parametrized by its curvilinear abscissa $M(s)$, $\frac{dM}{ds} = S$, its motion is then:

$$\frac{dM}{dt} = \frac{dM}{ds} \times \frac{d^2 M}{ds^2}$$

time evolution Banica, De la Hoz, Vega ...



Discrete time evolution Landau Lifshitz

- How to define the vertex?
- use lax matrices of XXX chain (6 vertex model)

$$l(u) = u + \mathrm{S}.\sigma = \begin{pmatrix} u + S^3 & S^- \\ S^+ & u + S^3 \end{pmatrix}$$

Discrete time evolution Landau Lifshitz

- How to define the vertex?
- use lax matrices of XXX chain (6 vertex model)

$$l(u) = u + \mathrm{S}.\sigma = \begin{pmatrix} u + S^3 & S^- \\ S^+ & u + S^3 \end{pmatrix}$$

- Solve for Darboux transform:

$$l(u, V_i)l(u-\tau, S_i) = l(u-\tau, V_{i+1})l(u, \bar{S}_i)$$

This time auxiliary space has the same lax matrix as for the dynamical variables.

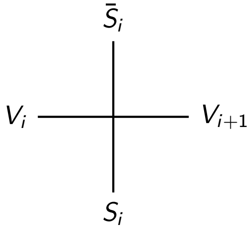
Discrete time evolution Landau Lifshitz

- Obtain the solution :

$$\bar{S}_j = \frac{1}{\sigma^2 + \tau^2}(\tau^2 S_j + \sigma^2 V_j - \tau S_j \wedge V_j)$$

$$V_{j+1} = \frac{1}{\sigma^2 + \tau^2}(\tau^2 V_j + \sigma^2 S_j - \tau V_j \wedge S_j)$$

- τ is the time step.



transport

[Marco Znidaric **PRL**, 2011], Tomaz Prosen

Transport of spin current in XXZ chain

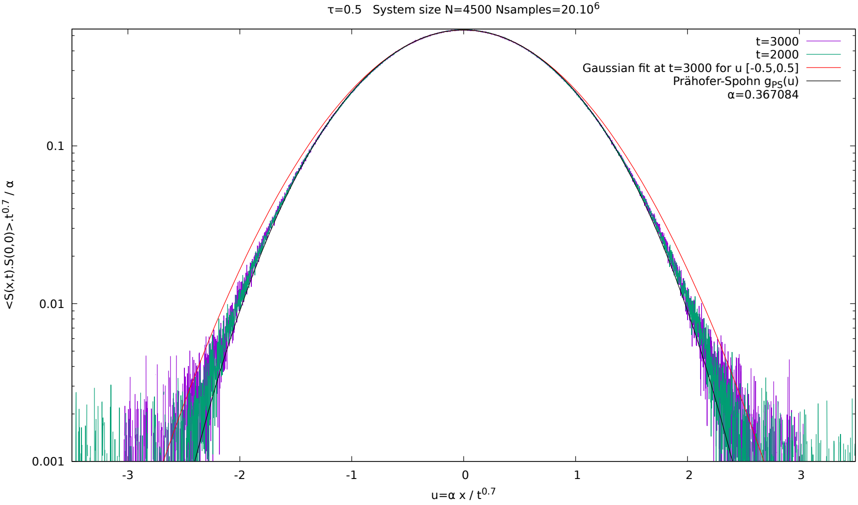
Infinite temperature transport:

- Ballistic for $\Delta < 1$
- Anomalous for $\Delta = 1$
- Diffusive for $\Delta < 1$

Infinite temperature transport Brownian filament at time zero.

[Ziga Krajnick, Tomaz Prosen **PRL**, 2019]

Spin correlations in Landau Lifshitz (Simulation by G. Misguich)



Thank you Rodney, you have given us so much