Universality in the Hyperbolic Plane

Bernard Nienhuis
Lorentz Institute, Leiden University
joint work with Daan Mulder

presented in honor and gratitude to Rodney J. Baxter

Hyperbolic plane

2D manifold with uniform negative curvature.
The sphere is a 2D manifold with uniform positive curvature.
Sphere can be mapped conformally to the (complex) plane by:

$$w = e^{i\phi} \tan \frac{\theta}{2}$$

likewise the hyperbolic plane can be mapped to the plane by

$$w = e^{i\phi} \tanh \frac{r}{2}$$

where r is the distance to the origin.

Somewhat counterintuitively, the finite sphere is mapped onto the infinite plane, while the infinite hyperbolic plane is mapped on a finite disk.

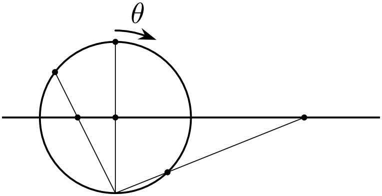
In 3D embedding, and conformal projection

the sphere is defined as

$$x^2 + y^2 + z^2 = 1$$

with metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$



Sphere projected on plane
(is conformal map)

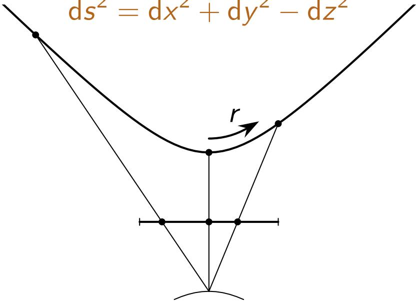
$$w = \mathrm{e}^{\mathrm{i}\phi} \tan \frac{\theta}{2}$$

and the hyperbolic plane as

$$x^2 + y^2 - z^2 = -1$$

with metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$$



Likewise for hyperbolic plane

$$w = \mathrm{e}^{\mathrm{i}\phi} \tanh \frac{r}{2}$$

In the hyperplane, the area of a disk with radius r is
$2\pi(\cosh(r) - 1)$
This grows (eventually) much faster than in the plane: $2\pi r^2$

Thinking of e.g. the random walk, this has many more ways to escape than in the euclidean plane.

Consider a discrete random walk with a growth constant, a weight for the probability to take a next step, rather than terminate.

When this weight is less than the inverse of the number of possibilities, the walk remains finite, and when it is more, it continues indefinitely.

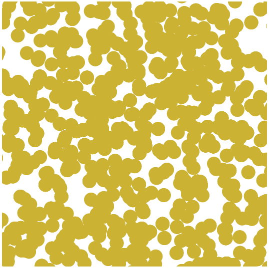
Percolation in the Euclidean plane

Uniform Poisson process sprinkling disks in the plane

percolation density p: Density of Poisson process $\times$ area of disks

Disks a and b are connected if they
(i) a and b overlap or
(ii) $\exists$ c, connected to a to b.

$$\exists p_c : \begin{cases} p \leq p_c \ \nexists & \text{infinite cluster} \\ p > p_c \ \exists! & \text{infinite cluster} \end{cases}$$



Largest cluster in domain of area A had on average $N(A)$ disks.
Then for large A

$$\begin{aligned} p < p_c : \quad & N(A) = \mathcal{O}(1) \\ p = p_c : \quad & N(A) = \mathcal{O}(A^{43/48}) \\ p > p_c : \quad & N(A) = \mathcal{O}(A) \end{aligned}$$

Percolation on the lattice:
the 'particles' have fixed positions on the vertices.

but the connection to neighbors is random (bond percolation)

Or the presence of a particle on a site is random (site percolation)

The exponent takes this value $\frac{43}{48}$ in the continuum, irrespective of the size or shape of the particles,

and also on the lattice, irrespective of the choice of lattice and of the choice between bond and site percolation.

Percolation in the hyperbolic plane

Two transitions:

$$\exists p_1, p_2 : \begin{cases} p < p_1 & \nexists \quad \text{infinite cluster} \\ p_1 < p < p_2 & \exists \quad \text{infinitely many infinite clusters} \\ p > p_2 & \exists! \quad \text{infinite cluster} \end{cases}$$

Suppose that again the central cluster in disk of area A

$$N(A) \propto A^{\psi}$$

Tobias Müller followed the abover procedure in the hyperplane, and proved that for large A this is indeed the case and that

$$\begin{array}{ll} p < p_1 : & \psi(p) = 0 \\ p_1 < p < p_2 : & \psi(p) \ \text{increases continuously from 0 to 1} \\ p > p_2 : & \psi(p) = 1 \end{array}$$

This attracted me to the subject.

It reminded me of the excitement we felt in the early 1970's when we learned of Baxter's solution of the 8-vertex model.

Just when had learned to accept the concept of universality: (critical exponents do not depend on the details of the interaction.)

Now we had a counter example.

Some panic in the community motivated a search to save the concept of universality.

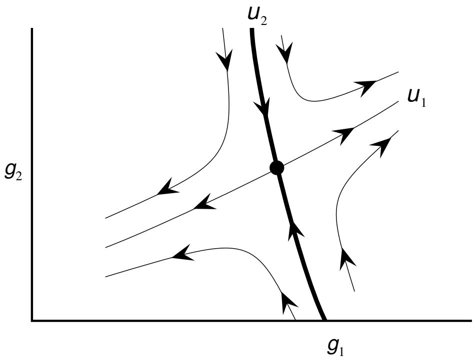
Eventually universality was saved, but limited:
In exceptional cases all critical exponents depend on only **one** parameter.

Why do we believe universality?

- Because we observe it.
- Because the theory of the Renormalization Group predicts it.

If there is a Renormalization transformation.

The critical behavior on a whole critical manifold is controled by a fixed point.



All emerging length scales are driven by the same correlation length. RG scales it, and when it is infinite, the system is scale free.

In the hyperbolic plane Renormalization is problematic:
Besides the correlation length, there is an intrinsic scale: the radius of curvature R_c.

Three scales are important:

- R_p the microscopic scale, e.g. the radius of the particles.
- R_c the radius of curvature
- R_d the radius of the domain

When $R_p \ll R_c$ a field theory as continuum limit is natural, perhaps at criticality a CFT.

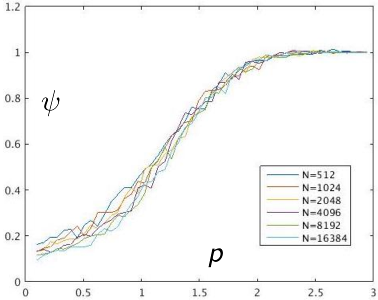
We work in the limit $R_p \approx R_c \ll R_d$.

This permits the use of regular lattices, which in the hyperbolic plane have a lattice distance of the order of R_c.
But before we tried a lattice we worked in the continuum and redid what Müller did.

Can we estimate $\psi(p)$?
Indeed we can.

It is a relatively boring function

The central cluster, is better than the largest one.



It is an exponent, so we could expect it to be universal, i.e. independent om microscopic details?

But the very definition of p depends on microscopic detail. (e.g. replace disks by line segments)

But suppose we have two exponents: $\psi_1(p)$ and $\psi_2(p)$, then we can eliminate p and find $\psi_2(\psi_1)$.
And this function could be universal.

First attempt at another exponent: the perimeter length of the central cluster.
Let the perimeter length of the central cluster: $L \propto A^{\psi_2}$

Unfortunately, $\psi_2(p > p_1) \approx 1$.

In retrospect not surprising: A disk-like cluster has perimeter $L = \sinh r$, and area $A = \cosh r - 1$, so $L = \mathcal{O}(A)$, i.e. exponent 1.
This is the same as for a stick-like cluster.
The exponent of a fractal cluster should be in between that of the disk and the stick.

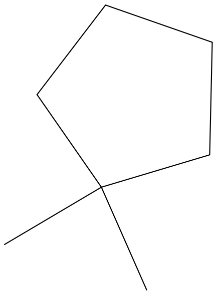
Second attempt: Radius of gyration R_g (spatial extent)
Not immediately obvious how to define and calculate.
We managed.
Again in the critical regime R_g virtually independent of p.
The cluster gets more massive, but also more compact.

A successful attempt for an alternative exponent, is the size of the central dual cluster.

The corresponding exponent, say ψ_d, must decrease from 1 to 0, as ψ increases from 0 to 1.

This is not difficult to define dual clusters in the continuum, but very difficult to measure. Therefore we switched to regular lattices to work with.

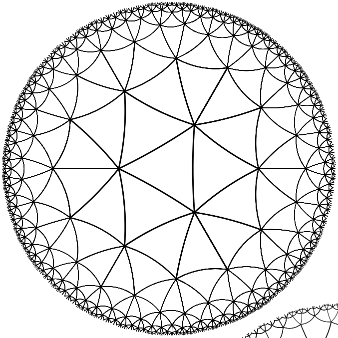
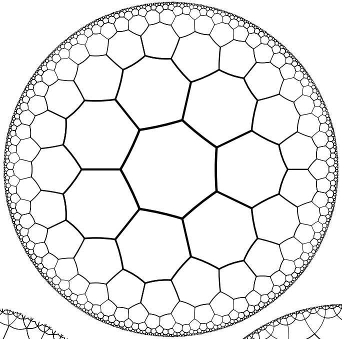
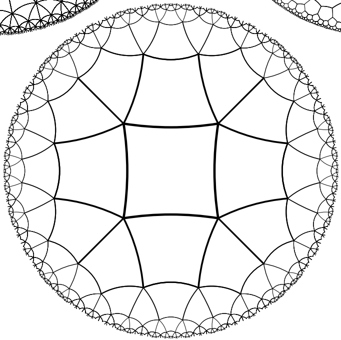
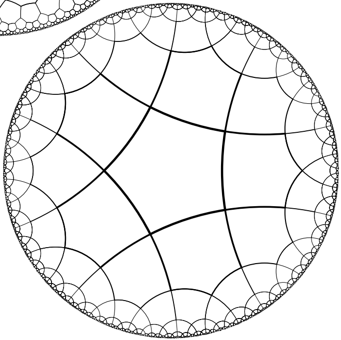
We define regular lattice to have equivalent vertices, faces and edges, i.e. each pair of vertices (or faces or edges) can be mapped into each other by an element of the lattice symmetry group.



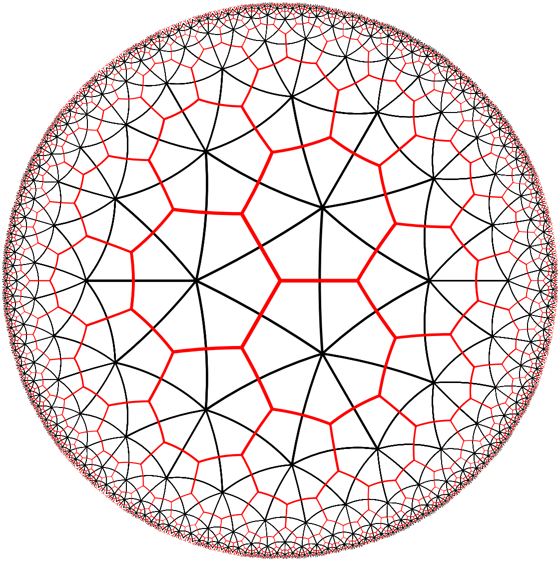
That leaves two numbers to specify: the number of edges around each face, say **f** and the number of edges incident on a vertex, say **v**.

Pairs $(f, v) \in \{(4,4), (3,6), (6,3)\}$ give the square, triangular and hexagonal lattice respectively.

The pairs $(f, v) \in \{(3,3),\ (3,4),\ (3,5),\ (4,3),\ (5,3)\}$ give the platonic solids, the tetrahedron, octahedron, icosahedron, cube, dodecahedron respectively. (the "$4v/(2v - vf + 2f)$-hedron")

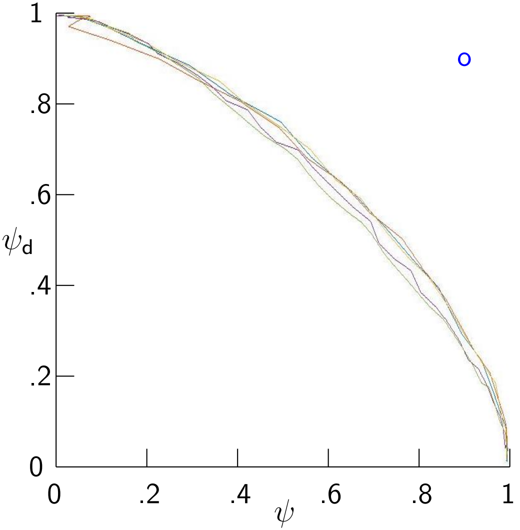





Just like in the Euclidean plane, interchanging $f \leftrightarrow v$ turns the lattice into its dual



ψ is the exponent relating the size of the central cluster to the arrea of the disk.

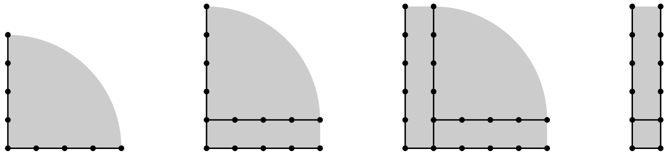
The corresponding exponent of the dual cluster, ψ_d, decreases from 1 to 0, as exponent ψ increases from 0 to 1.



The colored lines represent respectively the (3,7), (7,3), (5,5), (3,8), (8,3), (6,6), (3,9), (9,3) lattice.

The exponent for the Euclidean plane, marked by o is clearly not on the line.

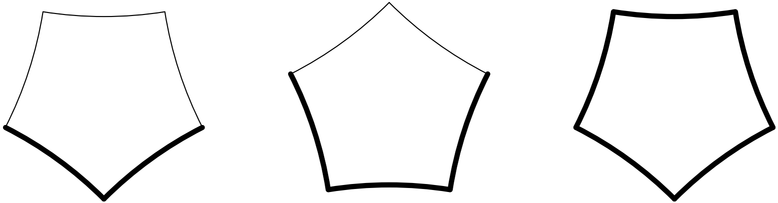
To improve the data we use a numerical approach to the Corner Transfer Matrix (another famous invention of Baxter).
In particular the Corner Transfer Matrix Renormalization Group. (Nishino and Okunishi).



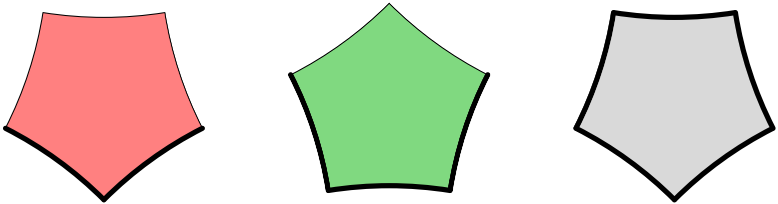
The CTM can be grown by successively multiplying it with a linear transfer matrix.
The linear transfer matrix must grow by adding two sites.

To make this computationally feasible, one projects on the eigenspaces of the, say M largest eigenvalues.

Application to a hyperbolic lattice:

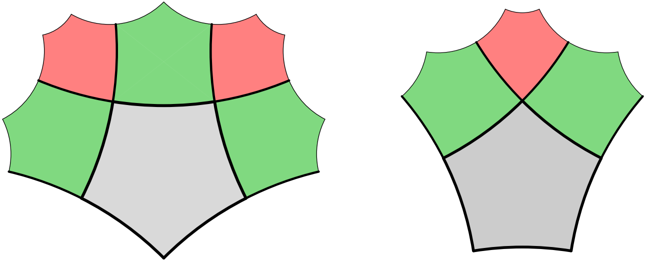


The pieces are merged at the fat edges, the thin edges represent the outer boundary.
They are colored to distinguish their role.

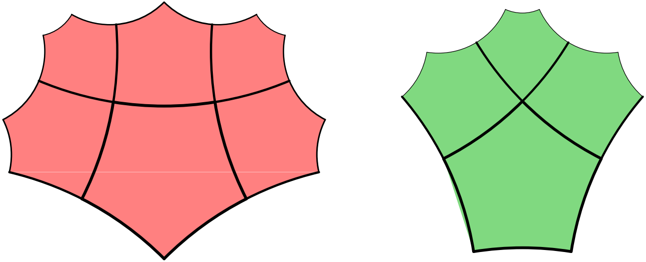


The red object is a very small CTM, or wedge. The green object is a generalization of the half open row-to-row transfer matrix, here called beam. The gray object is just a single face.

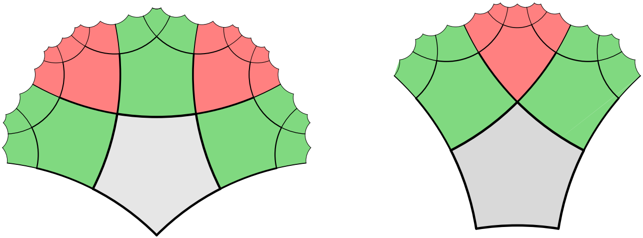
Now for the recursion:



Color the objects obtained:



and the next step



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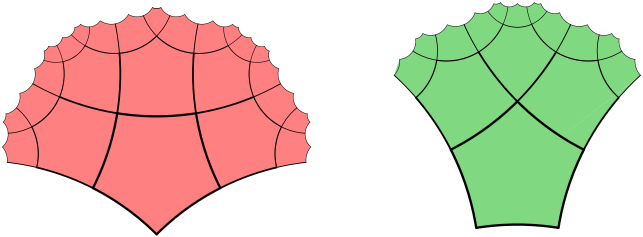
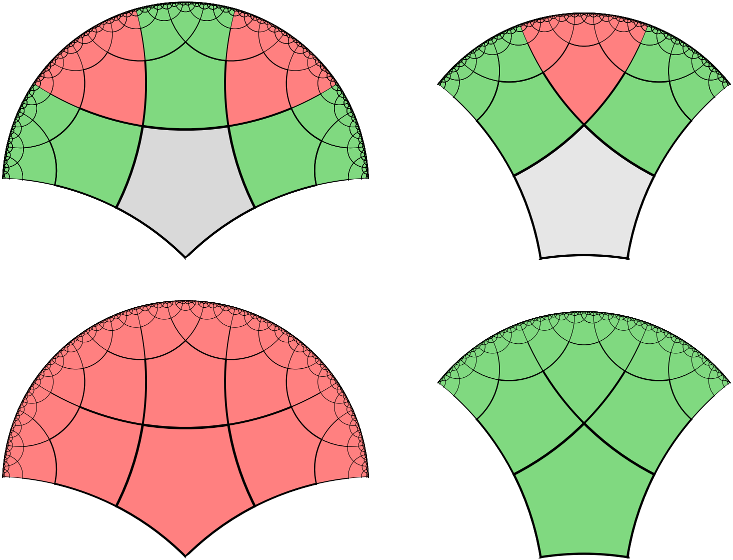


And continue this as far as possible



One difficulty is that while this way we can calculate the partition sum, the exponent ψ is not in the books. The size of the central cluster $N(A)$, can not be calculated in this way. Instead we need to focus on observables like the number of clusters, or cluster / dual-cluster boundaries that reach from the center of the disk to the boundary.

Another option is exponents associated with the number of cluster boundaries surrounding the center.

Work in progress

Concluding remark

Rodney Baxter has inspired me ever since 1973 when I learned of the 8-vertex model.

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