The Hausdorff dimension of badly approximable sets

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Number Theory Down Under

Motivation

Theorem (Dirichlet 1842)

For any $x = (x_1, x_2) \in [0,1]^2$ *and any* $N \in \mathbb{N}$ *there exists* $(p, q) = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{N}$ *solving*

$$\max_{i=1,2} \left| x_i - \frac{p_i}{q} \right| < \frac{1}{qN^{\frac{1}{2}}}, \quad 0 < q \leq N.$$

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Corollary

For any $x \in [0,1]^2$ *there exists infinitely many* $(p,q) \in \mathbb{Z}^2 \times \mathbb{N}$ *solving*

$$\max_{i=1,2} \left| x_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+\frac{1}{2}}}$$

Motivation

Let $\psi : \mathbb{N} \to \mathbb{R}_+$ *be a monotonic function decreasing to* 0. *Define*

$$\mathcal{W}_2(\psi) := \left\{ x \in [0,1]^2 : \max_{i=1,2} \left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q}, \quad \textit{for i.m.} \quad \frac{p}{q} \in \mathbb{Q}^2 \right\}$$

$$= \limsup_{q\to\infty} \bigcup_{0 \le p_1, p_2 \le q} B\left(\frac{p}{q}, \frac{\psi(q)}{q} \right).$$

For $\tau \in \mathbb{R}_+$ *write* $\psi_\tau(q) = q^{-\tau}$.

Define

$$\mathbf{Bad}_2 = \left\{ x \in [0,1]^2 : \begin{array}{c} \exists\, c(x) > 0 \ \ \max_{i=1,2} \left| x_i - \frac{p_i}{q} \right| > c(x) q^{-1-\frac{1}{2}} \\ \textit{for all } \frac{p}{q} \in \mathbb{Q}^2. \end{array} \right\}$$

$$= [0,1]^2 \backslash \bigcap_{k=1}^{\infty} \mathcal{W}_2\left(\frac{1}{k} \psi_{\frac{1}{2}} \right) = \mathcal{W}_2(\psi_{\frac{1}{2}}) \backslash \bigcap_{k=1}^{\infty} \mathcal{W}_2\left(\frac{1}{k} \psi_{\frac{1}{2}} \right).$$

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Question

How big is $\mathcal{W}_2(\psi)$*?*

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- (Khintchine 1926)

$$\lambda(\mathcal{W}_2(\psi)) = \begin{cases} 0 \text{ if } \sum_{q\in\mathbb{N}} \psi(q)^2 < \infty \\ 1 \text{ if } \sum_{q\in\mathbb{N}} \psi(q)^2 = \infty \,. \end{cases}$$

- (Jarnik 1929, Besicovitch 1934)
  $\dim_{\mathrm{H}} \mathcal{W}_2(\psi_\tau) = \frac{3}{1+\tau}$, $(\tau \geq \frac{1}{2})$.

- (Khintchine 1926)
  $\lambda(\mathbf{Bad}_2) = 0$.

- (Jarnik 1929, Schmidt 1960s)
  $\dim_{\mathrm{H}} \mathbf{Bad}_2 = 2$.

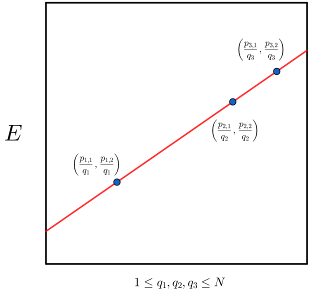
Motivation

Lemma (Simplex Lemma, Kristensen ,Thorn, Velani 2006)

Let $E \subset \mathbb{R}^2$ be a convex set with

$$\lambda(E) \leq (2!)^{-1} Q^{-3}.$$

Then the rational points $\frac{p}{q} \in \mathbb{Q}^2$ in E with $1 \leq q \leq Q$ lie on some line of $\mathbb{R}^2$.



Motivation

Lemma (Beresnevich, Velani 2006)

Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $[0,1]^2$ with $r(B_i)\to 0$ as $i\to\infty$. Suppose that

$$\lambda\left(\limsup_{i\to\infty} B_i\right) = 1\,.$$

Then for any ball $B\subset[0,1]^2$ and any $G>1$ there exists a finite disjoint subcollection $\mathcal{G}$ of $\{B_i\}_{i\geq G}$ contained in B such that

$$\lambda\left(B\cap\bigcup_{B_i\in\mathcal{G}} B_i\right)\geq\kappa\lambda(B)$$

for constant $\kappa>0$ independent of B and G.

ψ-badly approximable

Define

$$\mathbf{Bad}_2(\psi) = \left\{ x \in \mathbb{I}^2 : \exists c(x) > 0 \ \underbrace{c(x)\frac{\psi(q)}{q} \ \leq}_{\text{for all } \frac{p}{q} \in \mathbb{Q}^2} \max_{i=1,2} \left| x_i - \frac{p_i}{q} \right| \underbrace{< \ \frac{\psi(q)}{q}}_{\text{for i.m. } \frac{p}{q} \in \mathbb{Q}^2} \right\}$$

$$= \mathcal{W}_2(\psi) \backslash \bigcap_{k \in \mathbb{N}} \mathcal{W}_2\left(\frac{1}{k}\psi\right).$$

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$$= \mathcal{W}_2(\psi) \setminus \bigcap_{k \in \mathbb{N}} \mathcal{W}_2\left(\frac{1}{k}\psi\right).$$

Question

How big is $\mathbf{Bad}_2(\psi)$*?*

ψ-badly approximable

Contributions to dimension

- (Güting, 1963) For any $\varepsilon > 0$ and $\tau > 1$

$$\dim_H \mathcal{W}_1(\psi_\tau) \backslash \mathcal{W}_1(\psi_{\tau+\varepsilon}) = \dim_H \mathcal{W}_1(\psi_\tau).$$

- (Beresnevich, Dickinson, Velani, 2001) For any $\varepsilon > 0$ and $\tau > \frac{1}{2}$

$$\dim_H \mathcal{W}_2(\psi_\tau) \backslash \mathcal{W}_2((\log q)^{-\varepsilon} \psi_\tau) = \dim_H W_2(\psi_\tau)$$

- (Bugeaud, 2003) For any $1 > \varepsilon > 0$ and $\tau > 1$

$$\begin{aligned} \dim_H \mathcal{W}_1(\psi_\tau) \backslash \mathcal{W}_1(\varepsilon \psi_\tau) &= \dim_H \mathcal{W}_1(\psi_\tau) \\ \implies \quad &\dim_H \mathbf{Bad}_1(\psi_\tau) = \dim_H \mathcal{W}_1(\psi_\tau). \end{aligned}$$

ψ-badly approximable

Theorem (Koivusalo, Levesley, Zhang, W 2023)

For $\tau > \frac{1}{2}$

$$\dim_{\mathsf{H}} \mathbf{Bad}_2(\psi_\tau) = \dim_{\mathsf{H}} \mathcal{W}_2(\psi_\tau).$$

Sketch of proof

1. For small $c_N > 0$ construct a Cantor subset
$$\mathcal{C}(\tau, N) \subset [0,1]^2 \backslash \mathcal{W}_2(c_N \psi_\tau).$$
2. Construct a Cantor subset
$$\mathcal{D}(\tau) \subset \mathcal{W}_2(\psi_\tau) \cap \mathcal{C}(\tau, N).$$
3. Construct a measure μ on $\mathcal{D}(\tau)$ such that for any $\varepsilon > 0$ and any ball B with centre in $\mathcal{D}(\tau)$ and sufficiently small radius
$$\mu(B) \ll r(B)^{\dim_{\mathrm{H}} \mathcal{W}_2(\psi_\tau) - \varepsilon},$$
and so completing the proof by the mass distribution principle.

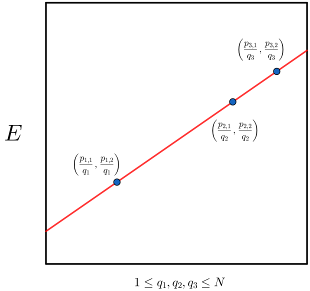
Constructing $\mathcal{C}(\tau, N)$

Lemma (Simplex Lemma)

Let $E \subset \mathbb{R}^2$ be a convex set with

$$\lambda(E) \leq (2!)^{-1} Q^{-3}.$$

Then the rational points $\frac{p}{q} \in \mathbb{Q}^2$ in E with $1 \leq q \leq Q$ lie on some line of $\mathbb{R}^2$.



Constructing $\mathcal{D}(\tau)$

Lemma ($T_{G,I}$ Lemma)

Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls with $r(B_i) \to 0$ as $i \to \infty$. Suppose that there exists constant $C > 0$ such that

$$\lambda\left(I \cap \limsup_{i\to\infty} B_i\right) \geq C\lambda(I)$$

for any $I \in \bigcup_{n\in\mathbb{N}} S_{\ell(n)}$. Then for any $I \in \bigcup_{n\in\mathbb{N}} S_{\ell(n)}$ and any $G > 1$ there is a finite subcollection $T_{G,I} \subset \{B_i : i \geq G\}$ such that the balls are disjoint, lie insides I, and

$$\lambda\left(\bigcup_{\tilde{B}\in T_{G,I}} \tilde{B}\right) \geq \kappa_1\lambda(I),$$

with κ_1 dependent on C.

Open problems

- $\mathbf{Bad}_2$ is winning on $[0,1]^2$. Is $\mathbf{Bad}_2(\psi_\tau)$ winning on $\mathcal{W}_2(\psi_\tau)$?
- Does the same result hold in the weighted setting? That is, does

$$\dim_{\mathrm{H}} \mathbf{Bad}_2(\psi_{\tau_1}, \psi_{\tau_2}) = \dim_{\mathrm{H}} \mathcal{W}_2(\psi_{\tau_1}, \psi_{\tau_2})?$$

- Under what conditions is it true that

$$\dim_{\mathrm{H}} \limsup_{i\to\infty} B_i = \dim_{\mathrm{H}} \limsup_{i\to\infty} B_i \setminus \limsup_{i\to\infty} cB_i?$$

Thank you!