LECTURE NOTES ON NON-ELLIPTIC FREDHOLM THEORY

Contents

1. Lecture 1: The scattering calculus on $\mathbb{R}^n$ introduced

Recall that the standard Hörmander pseudodifferential calculus $\Psi^m(\mathbb{R}^n)$ is the space of operators A given by quantizations of symbols $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ as follows:

$$A\phi(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)\phi(y)\, dy\, d\xi, \quad \phi \in \mathcal{S}(\mathbb{R}^n)$$
$$= (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{\phi}(\xi)\, d\xi. \quad (1.1)$$

Here the symbol class $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is the class of smooth functions $a(x,\xi)$ satisfying the standard symbol estimates

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}. \quad (1.2)$$

Our notation is that $D_{x_j} = -i\partial_{x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, and $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ is the 'Japanese bracket'.

The scattering calculus on $\mathbb{R}^n$, denoted $\Psi_{\mathrm{sc}}^{*,*}(\mathbb{R}^n)$, is a sub-calculus that is more symmetric in (x,ξ). Given two orders (m,l), the set of scattering symbols of order (m,l), denoted $S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n)$ are by definition the smooth functions $a(x,\xi)$ satisfying the estimates

$$\left| D_x^\alpha D_\xi^\beta a(x,\xi) \right| \le C_{\alpha\beta} \langle x \rangle^{l-|\alpha|} \langle \xi \rangle^{m-|\beta|}. \quad (1.3)$$

These symbol estimates endow $S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n)$ with the structure of a Fréchet space, with an increasing sequence of norms defined by

$$\|a\|_{m,l;k} = \sup_{(x,\xi)\in\mathbb{R}^{2n}, |\alpha|+|\beta|\le k} \left| D_x^\alpha D_\xi^\beta a(x,\xi) \langle x \rangle^{-l+|\alpha|} \langle \xi \rangle^{-m+|\beta|} \right|. \quad (1.4)$$

The quantization of such symbols gives us the space of scattering pseudodifferential operators of order (m,l), denoted $\Psi_{\mathrm{sc}}^{m,l}(\mathbb{R}^n)$.

The point of the scattering calculus is

- It is an ideal tool for doing scattering theory, i.e. the study of the generalized eigenfunctions of operators on $\mathbb{R}^n$ and their large-distance asymptotics;
- It is the right framework for realizing certain non-elliptic operators as Fredholm maps betwen suitable Sobolev-like spaces.

Some examples of scattering pseudodifferential operators:

- Constant coefficient differential operators of order m on $\mathbb{R}^n$ are scattering pseudodifferential operators of order $(m, 0)$. To give specific examples, the Laplacian $\Delta = \sum_j D_{x_j} D_{x_j}$, the wave operators $D_t^2 - \Delta$ and the Klein-Gordon operator $D_t^2 - \Delta - m^2$ (where $m > 0$ is constant) are scattering pseudodifferential operators. (Notice that we use the 'microlocal' convention that Δ is a positive operator.)
- More generally, pseudodifferential operators of order m in the Hörmander class with symbols depending only on ξ (thus, Fourier multipliers) are scattering pseudodifferential operators of order $(m, 0)$.
- Differential operators of order m with coefficients that are themselves symbols in x of order l are in $\Psi_{\mathrm{sc}}^{m,l}(\mathbb{R}^n)$.
- Suppose that $g = \sum_{ij} g_{ij}(x) dx_i dx_j$ is an 'asymptotically Euclidean' metric on $\mathbb{R}^n$, i.e.
$$g_{ij}(x) - \delta_{ij} \in S^{-1}(\mathbb{R}^n),$$
then the associated Laplace operator,
$$\Delta_g = \sum_{ij} \frac{1}{\sqrt{g(x)}} D_{x_i} g^{ij}(x) \sqrt{g(x)} D_{x_j}, \quad g = \det g_{ij}(x),$$
is in $\Psi_{\mathrm{sc}}^{2,0}(\mathbb{R}^n)$. (Here $g^{ij}(x)$ is the inverse matrix to $g_{ij}(x)$.)
- A non-example: differential operator with periodic (and non-constant) coefficients are in the Hörmander class, but not in the scattering class of pseudodifferential operators.

1.1. **Basic Properties of Symbols.**

- If $m' \le m$ and $l' \le l$ then $S_{\mathrm{sc}}^{m',l'}(\mathbb{R}^n \times \mathbb{R}^n) \hookrightarrow S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n)$ is continuous.
- $D_x^\alpha D_\xi^\beta$ maps $S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n) \to S_{\mathrm{sc}}^{m-|\beta|,l-|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n)$ continuously.
- Pointwise multiplication is continuous
$$S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n) \times S_{\mathrm{sc}}^{m',l'}(\mathbb{R}^n \times \mathbb{R}^n) \to S_{\mathrm{sc}}^{m+m',l+l'}(\mathbb{R}^n \times \mathbb{R}^n).$$
- Density. Let $S_{\mathrm{sc}}^{-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n) = \bigcap_{m,l} S_{\mathrm{sc}}^{m,l}(\mathbb{R}^n \times \mathbb{R}^n)$ denote the residual space of symbols. Then $S_{\mathrm{sc}}^{-\infty,-\infty}$ is not dense in $S_{\mathrm{sc}}^{m,l}$. However, if $a \in S_{\mathrm{sc}}^{m,l}$, then there exists a sequence $a_j \in S_{\mathrm{sc}}^{-\infty,-\infty}$ that is uniformly bounded in $S_{\mathrm{sc}}^{m,l}$ and which converges to a in the (slightly weaker) topology of $S_{\mathrm{sc}}^{m',l'}$ for any $m' > m$ and $l' > l$.
- Asymptotic summation: given a sequence of orders $(m_j, l_j), j \ge 0$ with $m_j \searrow -\infty$, $l_j \searrow -\infty$, and a sequence of scattering symbols $a_j \in S_{\mathrm{sc}}^{m_j,l_j}$, there exists $a \in S_{\mathrm{sc}}^{m_0.l_0}$ such that
$$a - \sum_{j=0}^{J-1} a_j \in S_{\mathrm{sc}}^{m_J,l_J}.$$
We call a an asymptotic sum of the a_j; it is unique modulo $S_{\mathrm{sc}}^{-\infty,-\infty}$.

1.2. **Compactification of phase space.** We define the radial compactification of $\mathbb{R}^n$ as follows: using polar coordinates, we can represent $\mathbb{R}^n \setminus \{0\}$ as $(0,\infty)_r \times S^{n-1}_{\hat{x}}$, where $r = |x|$ and $\hat{x} = x/|x|$. We then let $s = 1/r$, so that $[1,\infty)_r = (0,1]_s$. We compactify $(0,1]_s$ to $[0,1]_s$ and then take the product with $S^{n-1}_{\hat{x}}$; we have thus added a 'sphere at infinity'. Formally our compactification $\overline{\mathbb{R}^n}$ is given by

$$\overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup [0,1]_s \times S^{n-1}_{\hat{x}} / \sim,$$

where the equivalence relation $\sim$ identifies $(r,\hat{x})$ (when $r \geq 1$) and $(s,\hat{x})$ exactly when $r = 1/s$. Topologically $\overline{\mathbb{R}^n}$ is a closed ball.

The compactification of phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is then defined to be $\overline{T^*\mathbb{R}^n} := \overline{\mathbb{R}^n}_x \times \overline{\mathbb{R}^n}_\xi$. It is a compact manifold with corners of codimension 2.

We can reformulate the symbol estimates using the compactification $\overline{T^*\mathbb{R}^n}$. We note that $a \in S^{m,l}_{\mathrm{sc}}$ if and only if a is smooth on $(T^*\mathbb{R}^n)$, and if

$$\big(\langle x\rangle D_x\big)^\alpha \big(\langle \xi\rangle D_\xi\big)^\beta a \in \langle \xi\rangle^m \langle x\rangle^l L^\infty(T^*\mathbb{R}^n)$$

for all multi-indices (α,β). We then claim that $\langle x\rangle D_x$ and $\langle \xi\rangle D_\xi$ generate all smooth vector fields on $\overline{T^*\mathbb{R}^n}$ tangent to the boundary. Let's show this. We can write

$$\langle x\rangle D_{x_k} = \frac{1}{\langle x\rangle} D_{x_k} + \sum_j \frac{x_j}{\langle x\rangle} x_j D_{x_k}$$

and since $\frac{x_j}{\langle x\rangle}$ is a smooth function on $\overline{\mathbb{R}^n}$, we see that it is equivalent to requiring that repeated applications of $x_j D_{x_k}$ and $\xi_j D_{\xi_k}$ (as (j,k) range between 1 and n independently) to a remain in the fixed space $\langle \xi\rangle^m \langle x\rangle^l L^\infty(T^*\mathbb{R}^n)$.

We claim that the vector fields $x_j D_{x_k}$ on $\overline{\mathbb{R}^n}$ generate (over $C^\infty(\overline{\mathbb{R}^n})$) all smooth vector fields on $\overline{\mathbb{R}^n}$ tangent to the boundary. To see this we notice that such vector fields are homogeneous of degree zero under dilations $x \mapsto ax$, $a > 0$. In polar coordinates r, y, where $y = (y_1, \dots, y_{n-1})$ are angular coordinates, i.e. functions of $\hat{x}$, vector fields that are homogeneous of degree zero take the form

$$\sum_j b_j(y) D_{y_j} + c(y) r D_r.$$

Changing to $s = 1/r$ this reads

$$\sum_j b_j(y) D_{y_j} - c(y) s D_s,$$

and this is evidently smooth in (s,y) (which is a smooth coordinate system on $\overline{\mathbb{R}^n}$ near the boundary), and tangent to the boundary since the coefficient of D_s vanishes when $s = 0$. So the reformulation of the definition of the symbol space is

$$S^{m,l}_{\mathrm{sc}}(\mathbb{R}^n \times \mathbb{R}^n) = \Big\{ a \in \langle \xi\rangle^m \langle x\rangle^l L^\infty(\overline{T^*\mathbb{R}^n}) \mid V_1 \dots V_k a \in \langle \xi\rangle^m \langle x\rangle^l L^\infty(\overline{T^*\mathbb{R}^n}) \text{ for all smooth vector fields } V_1, \dots V_k \text{ on } \overline{T^*\mathbb{R}^n} \text{ tangent to the boundary, and all } k \in \mathbb{N} \Big\}. \tag{1.5}$$

It's clear from this formulation that $C^\infty(\overline{T^*\mathbb{R}^n})$ (or more precisely the set of restrictions of such functions to the interior $T^*\mathbb{R}^n$) is contained in $S^{0,0}_{\mathrm{sc}}$; we call these symbols *classical* symbols of order $(0,0)$ and denote them $S^{0,0}_{\mathrm{sc,cl}}(T^*\mathbb{R}^n)$. We similarly define classical symbols of order (m,l) to be $S^{m,l}_{\mathrm{sc,cl}}(T^*\mathbb{R}^n) := \langle\xi\rangle^m\langle x\rangle^l S^{0,0}_{\mathrm{sc,cl}}(T^*\mathbb{R}^n)$. Classical symbols have a Taylor-like series at both spatial and frequency infinity. Choose boundary defining functions $s = 1/r$ for the boundary at spatial infinity and $\rho = 1/|\xi|$ at frequency infinity. (A boundary defining function for a boundary hypersurface H of a manifold with corners is a smooth function that is nonnegative, vanishes at H and whose differential is nonzero at H, i.e. it vanishes simply at H.) Then $a \in S^{m,l}_{\mathrm{sc,cl}}$ iff a admits asymptotic expansions as $s \to 0$ and as $\rho \to 0$ of the form

$$a \sim \sum_{j=0}^{\infty} s^{-l+j} a_j(\hat{x},\xi) \text{ where } |\xi| \text{ is bounded },\ a_j \in C^\infty, s \to 0, \tag{1.6}$$

as well as

$$a \sim \sum_{k=0}^{\infty} \rho^{-m+k} b_k(x,\hat{\xi}) \text{ where } |x| \text{ is bounded },\ b_k \in C^\infty, \rho \to 0. \tag{1.7}$$

Near the corner, we have a Taylor-like expansion in two variables (ρ, s):

$$a \sim \sum_{j,k=0}^{\infty} s^{-l+j}\rho^{-m+k} c_{jk}(\hat{x},\hat{\xi}) c_{jk} \in C^\infty, \rho \to 0,\ s \to 0. \tag{1.8}$$

Not all symbols are classical! A simple example of a non-classical symbol in $S^{0,0}_{\mathrm{sc}}$ is $\langle x\rangle^{i\lambda}$ or $\langle\xi\rangle^{i\lambda}$ where $\lambda \neq 0 \in \mathbb{R}$. Clearly, this symbol does not have a limit at the boundary of phase space, as classical symbols of order $(0,0)$ do.

1.3. **Exercises.**

Problem 1.1.
Prove the statements about density of symbols above. To prove the density statement, take a function $\chi \in C^\infty_c(\mathbb{R}^n)$, such that $\chi = 1$ on $B(0,1)$ and $\chi = 0$ outside $B(0,2)$. Let $a_j(x,\xi)$ be the function

$$a_j(x,\xi) = \chi(x/j)\chi(\xi/j)a(x,\xi)$$

and prove the properties stated above for the sequence a_j.

Problem 1.2. Prove the statement about asymptotic completeness above. In this case, we may construct a as a sum of the form

$$a = \sum_{j=0}^{\infty} a_j(x,\xi)\big(1 - \chi(\epsilon_j x)\big)\big(1 - \chi(\epsilon_j \xi)\big),$$

where the ϵ_j are a sequence of small parameters converging to zero sufficiently fast. Notice that this sum is locally finite for any fixed (x,ξ) since only finitely many terms are nonzero, so the sum converges to a smooth function. However, we need to show more, we need convergence in the symbol space. Assume, mostly for ease of notation, that $m_j = m - j$ and $l_j = l - j$.

(i) Prove that a sufficient condition for convergence, both of the series above to a in the topology of $S_{\mathrm{sc}}^{m,l}$, and of the series $a-\sum_{j=0}^{J} a_j$ in the topology of $S_{\mathrm{sc}}^{m-J,l-J}$, is that for all $k \geq 0,\ J \geq 0$,

$$\|a_{J+k}(x,\xi)\big(1-\chi(\epsilon_{J+k}x)\big)\big(1-\chi(\epsilon_{J+k}\xi)\big)\|_{m-J,l-J;J} \leq 2^{-k}.$$

(ii) Show that these conditions are satisfied given a finite number of smallness conditions on ϵ_j for each j. Hence we can satisfy all conditions and thus construct the asymptotic sum.

2. Lecture 2: Composition, principal symbol, ellipticity

2.1. **Composition.** As the term 'calculus' suggests, the scattering pseudodifferential calculus is a bigraded algebra, i.e.

$$\Psi_{\mathrm{sc}}^{m.l} \circ \Psi_{\mathrm{sc}}^{m',l'} \subset \Psi_{\mathrm{sc}}^{m+m',l+l'}. \tag{2.1}$$

We will give a partial proof of this fact, taking for granted the graded algebra property of the Hörmander calculus. First we treat the case $l \leq 0, l' \leq 0$; it is easy to reduce to this case (see the first exercise). Let $A \in \Psi_{\mathrm{sc}}^{m,l}$ and $B \in \Psi^{m',l'}$. Then $A \in \Psi^m$ and $B \in \Psi^{m'}$ (the usual Hörmander calculus), so we know that $A \circ B \in \Psi^{m+m'}$. We need to show that the symbol of $A \circ B$ is in the space $S_{\mathrm{sc}}^{m+m',l+l'}$, which is a stronger condition. Let's consider the first the asymptotic expansion of the symbol of the composition:

$$\sigma_L(A \circ B) \sim \sum_{\alpha} \frac{\big(D_\xi^\alpha a_L(x,\xi)\big)\big(\partial_x^\alpha b_L(x,\xi)\big)}{\alpha!} \tag{2.2}$$

Consider all terms with $|\alpha| = j$. We have

$$D_\xi^\alpha a_L(x,\xi) \in S_{\mathrm{sc}}^{m-|\alpha|,l}, \quad \partial_x^\alpha b_L(x,\xi) \in S_{\mathrm{sc}}^{m',l'-|\alpha|}$$

so taking the pointwise product gives a symbol in $S_{\mathrm{sc}}^{m+m'-|\alpha|,l+l'-|\alpha|}$ for all such terms. We can asymptotically sum such terms since both exponents are tending to $-\infty$ as $|\alpha| \to \infty$, obtaining a symbol in $S_{\mathrm{sc}}^{m+m,l+l'}$. This is a good sign, but is not a proof as we also have to consider the properties of the remainder term.

There are three main approaches to composition: Gauss transforms, used in Hörmander's treatise [4], as well as Zworski [16], Dimassi-Sjöstrand [1], etc; left-right reduction, used by lecture notes of Melrose, Vasy [13] and Hintz [3]; and using pullback and pushforward of distributions via double and triple spaces, due to Melrose (e.g. [6]). We will follow the left/right reduction method. This involves quantizing symbols that depend on both the left and right variables of the Schwartz kernel of the operator, i.e. we quantize symbols $a(x,y,\xi)$ in a certain symbol class to the operator with Schwartz kernel

$$K_a(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi. \tag{2.3}$$

The relevant class here is symbols that behave in a scattering-like manner in the x and y variables independently: that is, we consider classes $S_{\mathrm{sc}}^{m,l,\tilde{l}}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n)$ satisfying product-type estimates

$$\Big|D_x^\alpha D_y^{\tilde{\alpha}} D_\xi^\beta a(x,\xi)\Big| \leq C_{\alpha\beta}\langle x\rangle^{l-|\alpha|}\langle y\rangle^{\tilde{l}-\tilde{\alpha}}\langle \xi\rangle^{m-|\beta|}. \tag{2.4}$$

It turns out that these quantize to elements of $\Psi_{\mathrm{sc}}^{m,l+\tilde{l}}$. That is, only the sum $l+\tilde{l}$ is relevant to the spatial order of the operator, which is not surprising as only the jet of the symbol $a(x,y,\xi)$ on the diagonal $x=y$ affects the operator (modulo residual terms). This is shown in Vasy's Grenoble lecture notes [].

To show composition, then, we follow the following standard strategy:

- We show that each symbol $a(x,y,\xi) \in S_{\mathrm{sc}}^{m,l,\tilde{l}}$ can be left or right reduced in a unique manner. That is, there are uniquely determined symbols $a_L(x,\xi)$ and $a_R(y,\xi)$, both in $S_{\mathrm{sc}}^{m,l+\tilde{l}}$, that quantize to the same operator as (2.3).
- We observe that the composition $A \circ B$ is a pseudodifferential operator with symbol $a_L(x,\xi)b_R(y,\xi)$.
- We then perform, say, left reduction on this symbol to show that we obtain something in $S_{\mathrm{sc}}^{m+m',l+l'}$. One expands the symbol in a Taylor series around the diagonal $x=y$. As we have already seen, the terms in the asymptotic expansion are no problem, but we need to consider the remainder term after subtracting the first J terms in the expansion. There is a subtlety, as this remainder term is obtained from the remainder term in the Taylor expansion of $a_L(x,\xi)b_R(y,\xi)$ at $x=y$, and involves an integral along the line segment from x to y (the usual Taylor formula integral remainder term). The problem is that if $|x|$ is large, and y is approximately $-x$, then the line segment passes near the origin. It means that the remainder might, ostensibly, decay much less than the terms in the expansion, since spatial decay happens when x and/or y are large. To circumvent this difficulty, we can decompose our symbol $a_L(x,\xi)b_R(y,\xi)$ into a part far from the diagonal and a part near the diagonal. The part far from the diagonal produces a residual operator as is shown by standard integration-by-parts tricks – see exercises. The part near the diagonal has the good property that the line segment from x to y has distance from the origin comparable to $|x|$ and $|y|$, and allows us to prove uniform decay of the remainder term as required. See Vasy's Grenoble notes [13] for the details.

We obtain the bigraded algebra property of the scattering calculus, and we have also confirmed that the symbol of the composition admits the usual asymptotic expansion (2.2), which is an *even better expansion* than the usual expansion in the sense that we gain spatial decay as well as frequency decay with each additional term. Indeed, in the scattering calculus, if we write R_N for the remainder term of expansion of $A \circ B$, then $R_N \in \Psi_{\mathrm{sc}}^{m+m'-N,l+l'-N}$ and every norm of R_N in this space (by which we mean the norms of $\sigma_L(R_N)$ in $S_{\mathrm{sc}}^{m+m'-N,l+l'-N}$) is bounded by a constant times a suitable norm of A in $\Psi_{\mathrm{sc}}^{m,l}$ and a suitable norm of B in $\Psi_{\mathrm{sc}}^{m',l'}$.

2.2. **Principal symbol.** Let $A = \mathrm{Op}_L(a_L) \in \Psi_{\mathrm{sc}}^{m,l}$. The principal symbol, $\sigma_{\mathrm{pr}}^{m,l}(A)$, is defined to be the equivalence class of a_L in $S_{\mathrm{sc}}^{m,l}/S_{\mathrm{sc}}^{m-1.l-1}$. It is the same as the equivalence class of a_R if $A = \mathrm{Op}_R(a_R)$.

Notice the difference with the Hörmander calculus: the principal symbol determines the leading behaviour of the symbol *both* as $|\xi| \to \infty$ for all x, and as $|x| \to \infty$ for all ξ (even $\xi = 0$).

For classical symbols, we can be more explicit. Assume that $(m,l) = (0,0)$ for convenience. Then a classical symbol $a \in S^{0,0}_{\mathrm{sc,cl}}$ is by definition C^∞ on $\overline{T^*\mathbb{R}^n}$, and in particular has a limit at $\partial\overline{T^*\mathbb{R}^n}$. On the other hand, if $b \in S^{-1,-1}_{\mathrm{sc}}$, then $b = s\rho\tilde{b}$, where $\tilde{b} \in C^\infty(\overline{T^*\mathbb{R}^n})$ where $s = \langle x\rangle^{-1}, \rho = \langle \xi \rangle^{-1}$. Such symbols are *precisely* those symbols of order $(0,0)$ that *vanish* at $\partial\overline{T^*\mathbb{R}^n}$. Thus, the principal symbol of $a \in S^{0,0}_{\mathrm{sc,cl}}$ can be identified with the boundary value of a on $\partial\overline{T^*\mathbb{R}^n}$. The boundary value of a is exactly knowing the limit of a as $|\xi| \to \infty$ for all x and $\hat{\xi}$, and the limit as $|x| \to \infty$ for all ξ and $\hat{x}$. This boundary value is 'smooth' in the sense that it is smooth on each boundary hypersurface, and is consistent at the corner. Next lecture, we will see that there is an important symmetry between these two boundary hypersurfaces of the manifold with corners $\overline{T^*\mathbb{R}^n}$.

Fundamental properties of the principal symbol:

- By definition of the principal symbol there is a short exact sequence:
$$0 \to \Psi^{m-1,l-1}_{\mathrm{sc}} \to \Psi^{m,l}_{\mathrm{sc}} \to S^{m,l}_{\mathrm{sc}}/S^{m-1,l-1}_{\mathrm{sc}} \to 0, \tag{2.5}$$
where the third arrow is the principal symbol map.
- The principal symbol is multiplicative:
$$\sigma^{m+m',l+l'}_{\mathrm{pr}}(A \circ B) = \sigma^{m,l}_{\mathrm{pr}}(A)\sigma^{m',l'}_{\mathrm{pr}}(B). \tag{2.6}$$
- We also have the formula for the principal symbol of a commutator $[A,B] = AB - BA$. Recall some background to this identity: the cotangent bundle of any manifold, here $\mathbb{R}^n$, is a symplectic manifold with a canonically defined symplectic form $\omega = \sum_j d\xi_j \wedge dx_j$. This defines a Poisson bracket $\{\cdot,\cdot\}$ and for any real function p on $T^*\mathbb{R}^n$, a Hamilton vector field H_p on $T^*\mathbb{R}^n$ defined invariantly by
$$\omega(H_p, V) = dp(V) = V(p).$$
Then we have
$$\sigma^{m+m'-1,l+l'-1}_{\mathrm{pr}}(i[A,B]) = \{a,b\} = H_a(b) = -H_b(a), \quad a = \sigma^{m,l}_{\mathrm{pr}}(A),\ b = \sigma^{m',l'}_{\mathrm{pr}}(B). \tag{2.7}$$

2.3. **Ellipticity.** We say that $A \in \Psi^{m,l}_{\mathrm{sc}}$ is (totally) elliptic if its principal symbol is invertible,in the sense that there exists $B_0 \in \Psi^{-m,-l}_{\mathrm{sc}}$ such that
$$\sigma^{m,l}_{\mathrm{pr}}(A) \cdot \sigma^{-m,-l}_{\mathrm{pr}}(B_0) = 1 \in S^{0,0}_{\mathrm{sc}}/S^{-1,-1}_{\mathrm{sc}}.$$
In other words,
$$\sigma_L(A) \cdot \sigma_L(B_0) - 1 \in S^{-1,-1}_{\mathrm{sc}}. \tag{2.8}$$

NB: this is a very strong version of ellipticity! The standard Laplacian Δ is *not* elliptic according to this definition, for example: although the principal symbol at frequency infinity is elliptic, the principal symbol at spatial infinity is not; in fact, it vanishes at $\xi = 0$. However, $\Delta + \lambda$ is totally elliptic in the scattering calculus provided $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$.

The standard construction, adapted to the scattering calculus, shows that if A is an elliptic scattering pseudodifferential operator, then there is an inverse modulo residual operators in the scattering calculus. Let us show this.

Definition 2.1. *Let $A \in \Psi_{\mathrm{sc}}^{m,l}$ be elliptic. A parametrix for A is an operator B such that*

$$AB - \mathrm{Id} \in \Psi_{\mathrm{sc}}^{-\infty,-\infty} \ni BA - \mathrm{Id}. \tag{2.9}$$

Proposition 2.1. *Every elliptic element of the scattering calculus has a parametrix.*

Proof. Given $A \in \Psi_{\mathrm{sc}}^{m,l}$ elliptic, choose $B_0 \in \Psi_{\mathrm{sc}}^{-m,-l}$ as above. From (2.8), we see that

$$AB_0 - \mathrm{Id} = -R_1 \in \Psi_{\mathrm{sc}}^{-1,-1}. \tag{2.10}$$

We want to invert $\mathrm{Id} - R_1$. Modulo residual operators we can do this with a Neumann series: let R be an asymptotic sum

$$R \sim R_1 + R_1^2 + R_1^3 + \dots,$$

which exists as $R_1^j \in \Psi_{\mathrm{sc}}^{-j,-j}$. Then we have

$$(\mathrm{Id} - R_1)(\mathrm{Id} + R) = \mathrm{Id} + R_r, \quad R_r \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}.$$

So defining $B_r = B_0(\mathrm{Id} + R)$ we have

$$AB_r = \mathrm{Id} + R_r.$$

In the same way, putting B_0 on the left of A instead of to the right, we can find B_l such that

$$B_l A = \mathrm{Id} + R_l, \quad R_l \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}.$$

We then note that B_l and B_r differ by a residual operator. This follows from the following calculation:

$$B_l A B_r = B_r + R_l B_r = B_l + B_l R_r.$$

It follows that either B_l or B_r is a parametrix for A. □

We remark that for classical symbols of order $(0,0)$, ellipticity is equivalent to the condition that $a\big|_{\partial\overline{T^*\mathbb{R}^n}}$ is everywhere nonzero. Since any classical a is continuous on $\partial\overline{T^*\mathbb{R}^n}$, and $\partial\overline{T^*\mathbb{R}^n}$ is compact it follows that if a is classical and elliptic, then a is bounded away from zero on $\partial\overline{T^*\mathbb{R}^n}$ and hence a^{-1} is also smooth. By extending a^{-1} into the interior arbitrarily as a smooth function, we obtain an symbol b satisfying $ab - 1 \in S_{\mathrm{sc,cl}}^{-1,-1}$ so quantizing b leads to an operator B_0 as above.

For a general operator A, we define the elliptic and characteristic sets of A, which are subsets of $\partial\overline{T^*\mathbb{R}^n}$. This is easiest to do in the case of classical operators. If $A \in \Psi_{\mathrm{sc}}^{m,l}$ is classical, let $\tilde{a} = \langle\xi\rangle^{-m}\langle x\rangle^{-l}\sigma_L(A)$. Then $\tilde{a}$ is smooth on $\overline{T^*\mathbb{R}^n}$ and we can define

$$\begin{aligned} \mathrm{Ell}(A) &= \{q \in \partial\overline{T^*\mathbb{R}^n} \mid \tilde{a}(q) \neq 0\}, \\ \mathrm{Char}(A) &= \{q \in \partial\overline{T^*\mathbb{R}^n} \mid \tilde{a}(q) = 0\}. \end{aligned} \tag{2.11}$$

More generally, for a non-necessarily-classical operator A, we say that a point $q \in \partial\overline{T^*\mathbb{R}^n}$ is in $\mathrm{Ell}(A)$ iff there exists a symbol $b \in S_{\mathrm{sc}}^{-m,-l}$ such that $\sigma_L(A) \cdot b - 1$ is in $S_{\mathrm{sc}}^{-1,-1}$ in a neighbourhood of $q \in \overline{T^*\mathbb{R}^n}$. More precisely, there is a smooth function χ on $\overline{T^*\mathbb{R}^n}$, identically 1 near q, such that $\chi(\sigma_L(A) \cdot b - 1)$ is in $S_{\mathrm{sc}}^{-1,-1}$. The characteristic set $\mathrm{Char}(A)$ is defined to be the complement of the elliptic set in $\partial\overline{T^*\mathbb{R}^n}$. Notice that the elliptic set is an open subset of $\partial\overline{T^*\mathbb{R}^n}$, and hence the characteristic set is closed.

2.4. **Mapping properties of residual operators.** We will discuss the boundedness properties of scattering operators more fully in Lecture 3. But here, we make an elementary observation about the mapping properties of residual operators. It is easy to see that a symbol a of order $(-\infty, -\infty)$ is precisely a Schwartz function on $\mathbb{R}^{2n}$. It follows that the Schwartz kernel of $\mathrm{Op}(a)$ is also a Schwartz function on $\mathbb{R}^{2n}$, as it involves taking the Fourier transform of a in ξ. Such Schwartz kernels map distributions to Schwartz functions, since if u_1 and u_2 are distributions, and $K(x, y)$ is Schwartz in $\mathbb{R}^{2n}_{x,y}$, then

$$u_2 \mapsto \big\langle u_2, \langle K(x, \cdot), u_1 \rangle \big\rangle, \quad u \in \mathcal{S}'(\mathbb{R}^n)_x,$$

is a continuous linear functional on $\mathcal{S}'(\mathbb{R}^n)$, and thus $\langle K(x, \cdot), u_1 \rangle$ is Schwartz. Thus, residual operators map distributions to Schwartz functions.

It follows that the null space of any elliptic operator A is Schwartz. In fact, if B is a parametrix, with $BA = \mathrm{Id} + R$, R residual, then $Au = 0$ implies $(\mathrm{Id} + R)u = 0$, so $u = -Ru$ is Schwartz. This has some consequences for essential self-adjointness of symmetric elliptic operators.

2.5. **Applications of the elliptic parametrix construction.** We give a few corollaries. Fix $V \in S^{-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. Recall that a *bound state* of a Schrödinger operator $\triangle + V$ on $\mathbb{R}^n$ is an L^2-eigenfunction with negative eigenvalue. Our discussion above immediately implies

Proposition 2.2. *Bound states are necessarily all Schwartz functions.*

Next, we give a spectral-theoretic application. Recall that a symmetric, densely defined unbounded operator on a Hilbert space is said to be *essentially self-adjoint* if its closure is self-adjoint. Then, the spectral theorem can be brought to bear. In particular, all eigenfunctions of an essentially self-adjoint operator have real eigenvalues.

To show that essential self-adjointness is subtle:

Example. *Fix $\varepsilon \in \{-1, +1\}$ and $k \in \mathbb{N}$. Consider $P = D_x^2 + \varepsilon x^k \in \Psi_{\mathrm{sc}}^{2,k}(\mathbb{R})$. This is certainly symmetric:*

$$\langle Pu, v \rangle_{L^2} = \langle u, Pv \rangle_{L^2} \tag{2.12}$$

*whenever $u, v \in H^2(\mathbb{R})$. **Q.** Is it essentially self-adjoint, when considered as an unbounded operator on $L^2(\mathbb{R})$ with domain $C_c^\infty(\mathbb{R})$? **A.***

- *If k is even and $\varepsilon = +1$, or if $k \in \{0, 1, 2\}$, then P is essentially self-adjoint.*
- *Otherwise, P is not essentially self-adjoint. In fact, for any complex number λ, there exists an L^2-eigenfunction of P with that eigenvalue. So, P is not essentially self-adjoint for* any *domain $\mathcal{D} \subseteq H^2(\mathbb{R})$. (That we took $\mathcal{D} = C_c^\infty(\mathbb{R})$ is not important.)*

Most of this is a consequence of some classical ODE theory (Liouville–Green, or JWKB), which tells us that there exist solutions with prescribed exponential behavior as $x \to \pm\infty$. For example, if $k \geq 4$ is even and ε is negative, then any solution of the ODE $Pu = \lambda u$ is asymptotic to a linear combination of

$$\frac{1}{|x|^{k/4}} e^{\pm \frac{2i}{k+2} |x|^{(k+2)/2}} \tag{2.13}$$

as $|x| \to \infty$. As long as $k \geq 3$, this lies in $L^2(\mathbb{R})$. If instead $k \geq 3$ is odd, then there exists a u with the asymptotic eq. (2.13) on one side of the real line and superpolynomial decay on the other. Either way, we have an L^2-eigenfunction with eigenvalue λ. Similar arguments apply to the remaining cases. (If $k = 0, 1, 2$, then the asymptotic eq. (2.13) is incorrect, receiving a correction from the λ term.)

The following reformulation is useful:

Lemma 2.1. *Fix $m, s \in \mathbb{R}$. Let $\mathcal{D}$ denote a subspace $C_{\mathrm{c}}^\infty(\mathbb{R}^n) \subseteq \mathcal{D} \subseteq H^{m,s}(\mathbb{R}^n)$. Suppose that $A \in \Psi_{\mathrm{sc}}^{m,s}$ is L^2-symmetric (eq. (2.12)). Then, A is essentially self-adjoint on L^2 with domain $\mathcal{D}$ if and only if*

$$\ker_{L^2}(A \pm i) = \{0\},$$

for both choice of signs.

This reduces questions of the essential self-adjointness of differential operators P to a concrete question about the well-posedness of the PDE $(P \pm i)u = f$.

Proof. If A is essentially self-adjoint, then it has real spectrum, so the non-trivial part of the lemma is that $\ker_{L^2}(A \pm i) = \{0\}$ is sufficient to conclude essential self-adjointness.

Note that if P is L^2-symmetric, then we in fact have $\langle Pu, v\rangle_{L^2} = \langle u, Pv\rangle_{L^2}$ for all $u \in \mathcal{S}$ and $v \in \mathcal{S}'$. Here, the L^2-inner product is understood distributionally:

$$\langle \phi, \psi\rangle_{L^2} = \begin{cases} \phi^*(\psi) & \phi \in \mathcal{S}' \\ \psi(\phi^*) & \psi \in \mathcal{S}'. \end{cases} \tag{2.14}$$

Indeed, if we fix $u \in \mathcal{S}$, then $\langle Pu, v\rangle_{L^2}, \langle u, Pv\rangle_{L^2}$ depend continuously on v with respect to the topology of $\mathcal{S}'$ and agree when $v \in \mathcal{S}$.

Deficiency index theory (see e.g. [RS72, Chp. VIII §2]]) says that the essential self-adjointness of A on $\mathcal{D}$ is equivalent to the range of $A \pm i$, acting on $\mathcal{D}$, being dense in L^2, for both choices of sign. Therefore, it sufficient to consider only $\mathcal{D} = C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Now,

$$\begin{aligned} \overline{\mathrm{Ran}_{C_{\mathrm{c}}^\infty(\mathbb{R}^n)}(A \pm i)} = L^2 &\iff \mathrm{Ran}_{C_{\mathrm{c}}^\infty(\mathbb{R}^n)}(A \pm i)^\perp = \{0\} \\ &\iff \{u \in L^2 : \langle (A \pm i)v, u\rangle_{L^2} = 0 \text{ for all } v \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)\} = \{0\} \\ &\iff \{u \in L^2 : \langle v, (A \mp i)u\rangle_{L^2} = 0 \text{ for all } v \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)\} = \{0\} \\ &\iff \underbrace{\{u \in L^2 : (A \mp i)u = 0\}}_{\ker_{L^2}(A \mp i)} = \{0\}. \end{aligned} \tag{2.15}$$

□

Why is it not obvious that $\ker_{L^2}(A \pm i) = \{0\}$? This is the statement that $\pm i$ is not an eigenvalue of A, but we saw in the example above that this can happen, e.g. for $A = \partial_x^2 + x^3$. What goes wrong if we try to apply to the unbounded operator A the usual argument that bounded symmetric operators have only real eigenvalues? If $u \in L^2$ satisfies $Au = \lambda u$, *and if P were bounded*, then

$$\lambda^* \|u\|^2 = \langle Au, u\rangle = \langle u, Au\rangle = \lambda \|u\|^2, \tag{2.16}$$

which implies $\lambda = \lambda^*$ if $u \neq 0$. The problem is that if A is not bounded, then we only know $\langle Au, v\rangle_{L^2} = \langle u, Av\rangle_{L^2}$ for "nice" u, v, and it may not be the case that an L^2-eigenfunction is nice. For example, for $A = \partial_x^2 + x^3$, the $\ker_{L^2}(A \pm i)$ consists of functions which are smooth but which do not decay particularly rapidly at infinity. They are in L^2, but if we try to carry out the proof that $\langle Au, u\rangle_{L^2} = \langle u, Au\rangle_{L^2}$, the integration-by-parts produces boundary terms. In fact,

$$\langle Au, u\rangle_{L^2} \neq \langle u, Au\rangle_{L^2} \tag{2.17}$$

for such u.

Proposition 2.3. *If $V \in S^{-\varepsilon}(\overline{\mathbb{R}^n})$ is real, then the operator $P = \triangle + V$ is essentially self-adjoint on L^2 with domain C_c^∞ .*

Proof. By the argument above, it suffices to show that any L^2-eigenfunction of P with eigenvalue $\pm i$ is sufficiently nice for $\langle Pu, u\rangle_{L^2} = \langle u, Pu\rangle_{L^2}$ to hold.

Note that $P \pm \lambda$ is an elliptic element of $\Psi_{\mathrm{sc}}^{2,0}$ if λ is non-real. Consequently, any element of its null space must be Schwartz. □

2.6. **Exercises.**

Problem 2.1. Show that the composition result above in the case $l \leq 0, l' \leq 0$ implies composition for arbitrary orders.

Problem 2.2. Consider a symbol in the case $S_{\mathrm{sc}}^{m,l,\tilde{l}}$ with the property that it vanishes whenever $|x - y| \leq \langle x\rangle/4$. Show that the quantization of such a symbol is residual. (Integrate by parts suitably.)

Problem 2.3. Suppose that the characteristic set of $A \in \Psi_{\mathrm{sc}}^{m,0}$ is disjoint from fiber infinity. Suppose that A is symmetric, meaning that $\langle Au, v\rangle_{L^2} = \langle u, Av\rangle_{L^2}$ for all $u, v \in \mathcal{S}$.

Show that A, with domain $C_c^\infty(\mathbb{R}^n)$, is essentially self-adjoint on $L^2(\mathbb{R}^n)$.

Problem 2.4. Suppose that $V \in C^\infty(\overline{\mathbb{R}})$ satisfies $\lim_{x\to-\infty} V(x) = 0$. Suppose that u is a bound state, with energy $E < 0$. Show that u is smooth everywhere and Schwartz on the left- half-line.

Problem 2.5. If $k > 0$ and $V \in \mathcal{S}(\mathbb{R}^n)$, then $\triangle + kx^2 + V$ is essentially self-adjoint acting on $C_c^\infty(\mathbb{R})$. Hint: a coordinate change from x to some

$$\tilde{x} = \begin{cases} x & (x \leq 1) \\ x^2 & (x \geq 2) \end{cases} \tag{2.18}$$

may be useful.

Problem 2.6. Consider the differential operator on the real line given by $P = -\partial_x^2 + 1$. A student makes the following argument:

> *P is an elliptic element of $\Psi_{\mathrm{sc}}^{2,0}(\mathbb{R})$, and so, via elliptic regularity, $Pu = 0 \Rightarrow u \in \mathcal{S}(\mathbb{R})$.*

But, $u(x) = e^x$ solves $Pu = 0$ and is not Schwartz. What mistake did the student make?

3. Lecture 3: Weighted Sobolev spaces

3.1. **Intertwining with the Fourier transform.** In the first lecture, we defined $\Psi_{\mathrm{sc}}^{0,0} = \Psi_{\mathrm{sc}}^{0,0}(\mathbb{R}^n)$, which is a certain proper subset

$$\Psi_{\mathrm{sc}}^{0,0} \subsetneq \Psi^0$$

of Hörmander's $\Psi^0 = \Psi^0(\mathbb{R}^n)$. Roughly, one should think of sc-pseudodifferential operators as those ordinary pseudodifferential operators with "good" behavior at infinity.

The purpose of this subsection is to provide another motivation for the sc-calculus, namely that it is, in some sense, the simplest pseudodifferential calculus that *treats momentum/frequency and position on equal footing*. This is an attractive feature of the sc-calculus not possessed by Hörmander's Ψ.

One can develop this theme in many ways. Here is one:

Proposition 3.1. *If $\mathcal{F}$ denotes the Fourier transform, then $\mathcal{F} \circ \Psi_{\mathrm{sc}}^{s,\ell} \circ \mathcal{F}^{-1} = \Psi_{\mathrm{sc}}^{\ell,s}$. That is,*

$$\mathcal{F} \circ A \circ \mathcal{F}^{-1} \in \Psi_{\mathrm{sc}}^{s,\ell} \tag{3.1}$$

for all $A \in \Psi_{\mathrm{sc}}^{s,\ell}$.

Remark 3.1. In contrast: $\mathcal{F} \circ \Psi^0 \circ \mathcal{F}^{-1}$ is not a subset of Ψ^0. (We will not prove this, but it follows via a similar computation to that below.)

Proof. Fix $A = \mathrm{Op}_L(a)$, $a \in S_{\mathrm{sc}}^{s,\ell}$. Let us compute the action of $\mathcal{F} \circ A \circ \mathcal{F}^{-1}$ on $\phi \in \mathcal{S}(\mathbb{R}^n)$. Directly plugging into the definition eq. (1.1),

$$\mathcal{F} \circ A \circ \mathcal{F}^{-1}\phi(\zeta) = (2\pi)^{-n} \int\int e^{ix\cdot(\xi-\zeta)} a(x,\xi)\phi(\xi)d\xi dx = \int K(\zeta,\xi)\phi(\xi)d\xi, \tag{3.2}$$

where

$$K(\zeta,\xi) = (2\pi)^{-n}\int e^{ix\cdot(\xi-\zeta)}a(x,\xi)dx. \tag{3.3}$$

To make this more recognizable, we can rename dummy variables:

$$K(x,y) = (2\pi)^{-n}\int e^{i\xi\cdot(y-x)}a(\xi,y)d\xi = (2\pi)^{-n}\int e^{i\xi\cdot(x-y)}a(-\xi,y)d\xi. \tag{3.4}$$

We now recognize this as $K_{\tilde{a}}(x,y)$, where $\tilde{a}(x,y,\xi) = a(-\xi,y)$. Let $R(\pi/2) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the 90° rotation

$$R(\pi/2) : (x,\xi) \mapsto (-\xi, x).$$

Then, $\tilde{a} = a \circ R(\pi/2)$. So,

$$\mathcal{F} \circ A \circ \mathcal{F}^{-1} = \mathrm{Op}_R(a \circ R). \tag{3.5}$$

The key point is that $a \circ R$ is a sc-symbol, specifically one in $S_{\mathrm{sc}}^{s,m}$, *because* $R(\pi/2)$ *is (obviously) a map*

$$R(\pi/2) : S_{\mathrm{sc}}^{s,m} \to S_{\mathrm{sc}}^{m,s},$$

owing to the fact that the estimates used to define $S_{\mathrm{sc}}^{0,0}$ treat position and frequency on equal footing. □

As part of the previous proposition, we proved:

Proposition 3.2. *If $A \in \Psi_{\mathrm{sc}}$, then, if a denotes the full left symbol of A, then $a \circ R$ is the full right symbol of A.*

Corollary 3.1. *The principal symbol of* $\mathcal{F}\circ A\circ\mathcal{F}^{-1}$ *is* $a\circ R$.

Proof. For either left or right quantization, we have $A=\mathrm{Op}_{L/R}(a)\Rightarrow a\in\sigma(A)$. So, this follows immediately from Proposition 3.2. □

Remark 3.2. For the reader acquainted with Fourier integral operators (FIOs), let us remark that the Fourier transform can be thought of as an FIO quantizing the symplectomorphism $R^{\pm 1}$.

Then (since the composition of FIOs is an FIO), and since pseudodifferential operators are FIOs whose underlying symplectomorphism (/canonical relation) is the identity, $\mathcal{F}\circ A\circ\mathcal{F}^{-1}$ should be an FIO for any $A\in\Psi_{\mathrm{sc}}$, whose underlying symplectomorphism is $R^{\pm1}\circ\mathrm{id}\circ R^{\mp1}=\mathrm{id}$, i.e. a pseudodifferential operator.

3.2. **Weighted Sobolev spaces.** The ordinary (L^2-based) Sobolev spaces $H^s(\mathbb{R}^n)=\Psi^{-s}(\mathbb{R}^n)L^2(\mathbb{R}^n)$ play a fundamental role in the ordinary microlocal analysis of PDEs. The analogous role in scattering theory is played by the (polynomially) weighted Sobolev spaces

$$H^{s,\ell}_{\mathrm{sc}}(\mathbb{R}^n)=L^2(\mathbb{R}^n)=\langle x\rangle^{-\ell}H^s(\mathbb{R}^n)\subset\mathcal{S}'. \tag{3.6}$$

These come with a natural Hilbertizable topology, namely that coming from H^s, such that multiplication by $\langle x\rangle^{-\ell}$ defines a continuous map $H^s\to H^{s,\ell}_{\mathrm{sc}}$.

Remark 3.3. Warning: different conventions exist for how $H^{\bullet,\bullet}_{\mathrm{sc}}$ should be indexed. We are following the convention that higher orders mean more regularity and more decay.

In this subsection, we will present a few basic facts about weighted Sobolev spaces. Proofs will only be provided when they differ in some essential way from the corresponding facts about ordinary Sobolev spaces.

Proposition 3.3. *The map* $\iota:\mathcal{S}\to(H^{s,\ell}_{\mathrm{sc}})^*$ *given by* $\iota(f)(u)=\int uf$ *extends boundedly to an isomorphism* $H^{-s,-\ell}_{\mathrm{sc}}\cong(H^{s,\ell}_{\mathrm{sc}})^*$.

Proof sketch. Follows from the duality $(L^2)^*\cong L^2$. □

Proposition 3.4 (Schwartz representation theorem)**.**
- $\bigcap_{s,\ell\in\mathbb{R}}H^{s,\ell}_{\mathrm{sc}}=\mathcal{S}$,
- $\bigcup_{s,\ell\in\mathbb{R}}H^{s,\ell}_{\mathrm{sc}}=\mathcal{S}'$.

Proof sketch. The first fact follows from Sobolev embedding. The second follows by dualizing the first. □

In the spirit of the previous subsection, let us remark:

Proposition 3.5. $\mathcal{F}:H^{s,\ell}_{\mathrm{sc}}\to H^{\ell,s}_{\mathrm{sc}}$.

Example. *Suppose* $M\subseteq\mathbb{R}^n$ *is a closed codimension* k*-submanifold of* $\mathbb{R}^n$*. Then, the Dirac-*δ *function* δ_M *lies in* $H^{s,\ell}_{\mathrm{sc}}$ *whenever* $s<-k/2$*. So,* $\mathcal{F}\delta_M\in H^{s,\ell}_{\mathrm{sc}}$ *whenever* $\ell<-k/2$.

For example, if $M=\{0\}$*, then* $\mathcal{F}\delta_M=1$*. We are claiming that* $1\in\langle r\rangle^{\ell}H^s$ *whenever* $\ell<-n/2$*, which is true.*

Proposition 3.6. *If $A \in \Psi_{\mathrm{sc}}^{s,\ell}$, then, for all $s', \ell' \in \mathbb{R}$, A maps*

$$A : H_{\mathrm{sc}}^{s',\ell'} \to H_{\mathrm{sc}}^{s'-s,\ell'-\ell}, \tag{3.7}$$

and does so boundedly.

So, if $s, \ell > 0$, then applying $A \in \Psi_{\mathrm{sc}}^{s,\ell}$ reduces regularity by s orders *and* reduces decay by ℓ orders. Similar statements apply if $s \leq 0$ or $\ell \leq 0$, switching "reduces" to "increases" where necessary.

The proposition is an easy corollary of the L^2-boundedness of ordinary ΨDOs:

Proof. We have a commutative diagram

$$\begin{tikzcd} H_{\mathrm{sc}}^{s,\ell} \arrow[r, "A"] \arrow[d, "\times\langle x\rangle^{\ell'}"'] & H_{\mathrm{sc}}^{s'-s,\ell'-\ell} \\ H^{s'} \arrow[r, "\langle x\rangle^{\ell'-\ell}A\langle x\rangle^{-\ell'}"'] & H^{s-s'} \arrow[u, "\times\langle x\rangle^{\ell-\ell'}"'] \end{tikzcd} \tag{3.8}$$

in which all of the maps except the top map are known to be bounded. (The reason we know that $\langle x\rangle^{\ell'-\ell}A\langle x\rangle^{-\ell'}$ maps $H^{s'} \to H^{s'-s}$ boundedly is that, since $\langle x\rangle^r \in \Psi_{\mathrm{sc}}^{0,r}$ for any r,

$$\langle x\rangle^{\ell'-\ell}A\langle x\rangle^{-\ell'} \in \Psi_{\mathrm{sc}}^{0,\ell'-\ell}\Psi_{\mathrm{sc}}^{s,\ell}\Psi_{\mathrm{sc}}^{0,-\ell'} \subseteq \Psi_{\mathrm{sc}}^{s,0} \subset \Psi^s, \tag{3.9}$$

and we already know, as part of the theory of Ψ, that elements of Ψ^s map $H^{s'} \to H^{s'-s}$ boundedly.)

So, A maps $H_{\mathrm{sc}}^{s',\ell'} \to H_{\mathrm{sc}}^{s'-s,\ell'-\ell}$, and does so boundedly. □

Remark 3.4. It is sometimes useful to note that $\mathrm{Op}(a)$ depends continuously on a in the sense that if $a_1, a_2, \dots \in S_{\mathrm{sc}}^{s,\ell}$ is some sequence such that $a_n \to a$ in $S_{\mathrm{sc}}^{s,\ell}$, then

$$\mathrm{Op}(a_n) \to \mathrm{Op}(a)$$

in $\mathcal{L}(H_{\mathrm{sc}}^{s',\ell'}, H_{\mathrm{sc}}^{s'-s,\ell'-\ell})$.

3.3. **The Rellich theorem.** Recall the *Rellich compactness theorem*, which states that if X is a closed manifold, then the inclusion $H^{m'}(X) \hookrightarrow H^m(X)$ is compact whenever $m < m'$.

Importantly, this fails when $X = \mathbb{R}^n$. Indeed, fix $u \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Then, for any nonzero $\mathbf{k} \in \mathbb{R}^n$, the translates $u_n = u(\bullet + n\mathbf{k})$ converge weakly to 0 in H^m for every $m \in \mathbb{R}$. (Why?) But, $\|u_n\|_{H^m}$ is constant, and therefore not converging to 0. So, $H^{m'}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is never compact.

Instead, the "correct" analogue of the Rellich compactness theorem on $\mathbb{R}^n$ is:

Proposition 3.7. *For any s, s', ℓ, ℓ', with $s < s'$ and $\ell < \ell'$, the embedding $H_{\mathrm{sc}}^{s',\ell'} \hookrightarrow H_{\mathrm{sc}}^{s,\ell}$ is compact.*

Proof Idea. The full result follows easily from the $m, s = 0$ case.

So, we want to show that if $\varepsilon > 0$, then the inclusion $H_{\mathrm{sc}}^{\varepsilon,\varepsilon} \hookrightarrow L^2$ is compact. In other words, we want to show that if $u_1, u_2, \dots$ is a sequence of elements of $H_{\mathrm{sc}}^{\varepsilon,\varepsilon}$ converging weakly to 0 in this space, then $\|u_n\|_{L^2} \to 0$.

Via the Rellich compactness of the inclusion $H^\varepsilon \hookrightarrow L^2$ on *compact manifolds* (say, the torus $\mathbb{R}^n/\Lambda\mathbb{Z}^n$ for Λ large) (the Rellich compactness theorem), it must be the case that $\|\chi u_n\|_{L^2} \to 0$ for any $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$. This means that the L^2-mass of the u_n is leaving every compact subset. But we want to say that the L^2-mass of the u_n is going to zero. So, what we need to rule out is that the L^2 mass of the u_n "escapes" to spatial infinity without decaying. Intuitively, it makes sense that this is ruled out by weak convergence in $H_{\mathrm{sc}}^{\varepsilon,\varepsilon}$; were u_n to escape, it would be expected to be possible to construct an adversarial $v \in H_{\mathrm{sc}}^{-\varepsilon,-\varepsilon} = (H_{\mathrm{sc}}^{\varepsilon,\varepsilon})^*$ such that $\langle v, u_n\rangle \not\to 0$.

Rather than argue along these lines, it is somewhat easier to use an alternative characterization of compact maps between Hilbertizable spaces, namely that the compact maps are those in the closure (under the operator norm) of the set of finite-rank operators. □

Proof. For each $R > 0$, consider the multiplication operator $M_R = \chi(x/R)$. By the Rellich compactness theorem, there exists a finite-rank operator $F_R : H_{\mathrm{sc}}^{\varepsilon,\varepsilon} \to L^2$ such that $\|M_R - F_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} \le 1/R$. Now,

$$\|1 - F_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} \le \|1 - M_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} + \|M_R - F_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} \le \|1 - M_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} + 1/R. \tag{3.10}$$

But, $\chi(\bullet/R)$ converges, as $R \to \infty$, to 1 in $S_{\mathrm{sc}}^{\varepsilon,\varepsilon}$, for every $\varepsilon > 0$. [Exercise.] It follows that $\|1 - M_R\|_{H_{\mathrm{sc}}^{\varepsilon,\varepsilon}\to L^2} \to 0$ as $R \to \infty$. □

Suppose that $A, A' \in \Psi_{\mathrm{sc}}^{0,0}$ have the same principal symbol. Then, it follows that

$$A - A' \in \Psi_{\mathrm{sc}}^{-1,-1}. \tag{3.11}$$

So, the difference $K = A - A'$ is compact acting on any fixed sc-Sobolev space. *In the sc-calculus, principal symbols capture operators modulo compact errors.* This is the single most important property that the sc-calculus has that Hörmander's Ψ does not — it is why Ψ_{sc} is as useful as it is when doing scattering theory.

3.4. **A quick review of abstract Fredholm theory.** It is a fact of life that many PDEs one cares about are not well-posed. For example, if $\triangle_g$ denotes the Laplace–Beltrami operator on a closed Riemannian manifold (M, g), then, for $\lambda \in \mathbb{R}$, the problem

$$\begin{cases} u \in H^2(M), \\ \triangle_g u - \lambda u = f \in L^2 \end{cases} \tag{3.12}$$

is only well-posed if λ is not an eigenvalue of $\triangle_g$.

After well-posedness, Fredholmness is the next best thing. Recall that if $\mathcal{X}, \mathcal{Y}$ are two Banach spaces and $P : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map, then P is said to be *Fredholm* if the following three conditions are all satisfied:

(i) P has closed range,
(ii) P has finite-dimensional null space,
(iii) P has finite-dimensional cokernel $\mathcal{Y}/P\mathcal{X} \cong (P\mathcal{X})^\perp$.

Roughly, being Fredholm means being invertible modulo a finite-dimensional obstruction.

Example. *If (M, g) is a closed Riemannian manifold, then, for any first-order differential operator Q with smooth coefficients, the operator $P = \triangle_g + Q$ is Fredholm as a map $H^m(M) \to H^{m-2}(M)$, for any $m \in \mathbb{R}$. Thus, every eigenvalue of $\triangle_g$ has finite multiplicity.*

Example. *On $\mathbb{R}^n$, if*

$$\triangle = \sum_{j=1}^{n} D_{x_j}^2 = -\sum_{j=1}^{n} \partial_{x_j}^2$$

denotes the positive semidefinite *Laplacian and $\lambda > 0$, then $\triangle + \lambda$ is Fredholm as a map $H^m(M) \to H^{m-2}(M)$, for any $m \in \mathbb{R}$.*

This is not true if $\lambda \leq 0$. Indeed, then $P = \triangle + \lambda$, considered as a map $H^2 \to L^2$, has range

$$\operatorname{range} P = \mathcal{F}\{u \in L^2 : (\xi^2 - \lambda)^{-1} u \in L^2(\mathbb{R}^n)\}$$

which is dense in L^2 but not all of L^2, and therefore not closed.

Moral. Unlike on closed manifolds, Fredholmness on non-compact spaces can depend on seemingly lower-order operators. The reason for "seemingly" in the previous sentence is that $\triangle \in \Psi_{\mathrm{sc}}^{2,0}$ and $+\lambda \in \Psi_{\mathrm{sc}}^{0,0}$; so in terms of differential order, it is true that $\triangle$ is higher-order, but in terms of *decay order*, they are both 0th order.

Let $P : \mathcal{X} \to \mathcal{Y}$ denote a bounded linear map. A *semi-Fredholm* estimate is one of the form

$$\|u\|_{\mathcal{X}} \leq C(\|Pu\|_{\mathcal{Y}} + \|\iota u\|_{\mathcal{Z}}) \tag{3.13}$$

for $\iota : \mathcal{X} \to \mathcal{Z}$ a compact injection from $\mathcal{X}$ to some other Banach space $\mathcal{Z}$, and $C > 0$ a constant (independent of u). This essentially means that we can control u in terms of Pu and some *weak* norm $\|\phi u\|_{\mathcal{Z}}$ of u.

Remark 3.5. If P had no null space, then we would have an estimate $\|u\|_{\mathcal{X}} \leq C\|Pu\|_{\mathcal{Y}}$. So, a semi-Fredholm estimate is almost as good, except we have to allow a "small" error on the right-hand side to accomodate the fact that P may have null space.

The following fact about Fredholm operators is "standard."

Proposition 3.8. *Fix Hilbert spaces $\mathcal{X}, \mathcal{Y}$. Then, the following are equivalent:*

(i) P is Fredholm.
(ii) P is invertible modulo a compact error. That is, there exists a bounded linear operator $Q : \mathcal{Y} \to \mathcal{X}$ such that

$$QP - \mathrm{id}_{\mathcal{X}}, PQ - \mathrm{id}_{\mathcal{Y}}$$

are compact.
(iii) (Semi-Fredholm estimates.) P and $P^* : \mathcal{Y} \to \mathcal{X}$ both satisfy a semi-Fredholm estimate.

Proof. The equivalence of (i), (ii) really is a standard fact, explained in many functional analysis texts, so we will focus on the equivalence of (iii) with the other two.

- (iii) $\Rightarrow$ (i): Suppose that $u_1, u_2, u_3, \ldots$ were some infinite orthonormal sequence of elements of $\ker P$. Then, $u_n \to 0$ weakly in $\mathcal{X}$. Since ι is compact, this implies $\|\iota u_n\|_{\mathcal{Z}} \to 0$. But this violates the semi-Fredholm estimate for P, which, when applied to u_n, says
$$1 = \|u_n\|_{\mathcal{X}} \lesssim \|\iota u_n\|_{\mathcal{Z}}. \tag{3.14}$$
So, the kernel of P has to be finite-dimensional.
To see that the range of P is closed: suppose that $u_1, u_2, u_3, \ldots$ are some infinite sequence of nonzero elements of $\mathcal{X}$ such that $Pu_j \to f$. Suppose we knew that the u_j (or a subsequence thereof) stayed within some large ball in $\mathcal{X}$. Then, by the Banach–Alaoglu theorem, we could pass to some subsequence converging weakly to some $u_\infty \in \mathcal{X}$. Assuming without loss of generality that passing to the subsequence is not necessary, $Pu_j \to Pu_\infty$ weakly. This implies $Pu_\infty = f$, so $f \in \operatorname{Ran} P$.
Of course, it need not be the case that u_j stay bounded in $\mathcal{X}$, since we always could add an arbitrarily large element of $\ker P$ to u_j without changing its image under P. So, we might as well take $u_j \in (\ker P)^\perp$. Now suppose that still $\|u_j\| \to \infty$. Let $\hat{u}_j = u_j / \|u_j\|$. Since these stay bounded in $\mathcal{X}$, by Banach–Alaoglu we can assume that they converge weakly to some $v \in (\ker P)^\perp$. Now, $P\hat{u}_j \to Pv$ weakly, but also $\|P\hat{u}_j\| = \|Pu_j\|/\|u_j\| \to 0$. So, we must have $Pv = 0$. On the other hand, $\iota u_j \to \iota v$ strongly in $\mathcal{Z}$. The semi-Fredholm estimate says
$$1 = \|\hat{u}_j\|_{\mathcal{X}} \lesssim \|P\hat{u}_j\|_{\mathcal{Y}} + \|\iota \hat{u}_j\|_{\mathcal{Z}}. \tag{3.15}$$
Since $\|P\hat{u}_j\|_{\mathcal{Y}} \to 0$, this means that $\|\iota\hat{u}_j\|_{\mathcal{Z}}$, and therefore $\|\iota v\|_{\mathcal{Z}}$, must eventually be bounded below by something positive. So, $v \neq 0$. But, the three facts $v \in (\ker P)^\perp$, $Pv = 0$, $v \neq 0$ are inconsistent.
So, the range of P is closed.
From this, it follows that the cokernel is isomorphic to $\ker P^*$ (see Problem 3.3. The argument above shows that the finite-dimensionality of $\ker P^*$ follows from the semi-Fredholm argument.
- (ii) $\Rightarrow$ (iii). Let $K = QP - \mathrm{id}_{\mathcal{X}}$, which we assume is a compact operator on $\mathcal{X}$. Then,
$$\|u\|_{\mathcal{X}} \leq \|QPu\|_{\mathcal{X}} + \|Ku\|_{\mathcal{X}} \lesssim \|Pu\|_{\mathcal{Y}} + \|Ku\|_{\mathcal{X}}. \tag{3.16}$$
Let $\mathcal{Z} = \ker K \oplus K\mathcal{X}$, endowed with the norm $\|(u,k)\|_{\mathcal{Z}} = \|u\|_{\mathcal{X}} + \|k\|_{\mathcal{X}}$. Let $\iota : \mathcal{X} \to \mathcal{Z}$ be defined by $\iota : u \mapsto (Ku, \Pi_{\ker K} u)$. Then,
$$\|u\|_{\mathcal{X}} \lesssim \|Pu\|_{\mathcal{Y}} + \|\iota u\|_{\mathcal{Z}}. \tag{3.17}$$
So, we get a semi-Fredholm estimate for P.
Applying the same argument to P^*, we conclude that it satisfies a semi-Fredholm estimate.

□

The operator Q in above is called a *parametrix* for P.

3.5. **Elliptic Operators are Fredholm.** Now the main result of this lecture:

Proposition 3.9. *If $A \in \Psi_{\mathrm{sc}}^{s,\ell}$ is elliptic, then it is Fredholm as a map $H_{\mathrm{sc}}^{s',\ell'} \to H_{\mathrm{sc}}^{s'-s,\ell'-\ell}$.*

Proof. Follows immediately from the elliptic parametrix construction, the characterization Proposition 3.8 of Fredholmess, and the compactness of the embedding $H_{\mathrm{sc}}^{-1,-1} \hookrightarrow L^2(\mathbb{R}^n)$ □

Consequently:

Proposition 3.10. *If $A \in \Psi_{\mathrm{sc}}^{s,\ell}$ is elliptic, then $\ker_{\mathcal{S}'} A$ is finite-dimensional.*

Proof. By the elliptic parametrix construction, $\ker_{\mathcal{S}'} A$ consists entirely of Schwartz functions. Thus, they lie in the kernel of A restricted to any individual sc-Sobolev space, say L^2. By the the previous proposition, the Fredholmness of A acting on L^2 implies that the kernel is finite-dimensional. □

Corollary 3.2. *Consider $P = \triangle + V$, $V \in S^{-\varepsilon}$. For each $E < 0$, there are at most finitely many bound states with that energy.*

3.6. **Problems.**

Problem 3.1. Prove Proposition 3.4.

Problem 3.2. Prove Proposition 3.5.

Problem 3.3.
(1) Show that if $P : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map between Hilbert spaces $\mathcal{X}, \mathcal{Y}$ with closed range, then $\mathcal{Y}/P\mathcal{X} \cong \ker P^*$.
(2) Is this true if we do not assume that P has closed range?

Problem 3.4. Consider $u(x) = e^x \sin(e^x)$.
(1) Show that $u \in H_{\mathrm{sc}}^{-1,\ell}(\mathbb{R})$ for l sufficiently negative.
(2) (Optional.) For which $H_{\mathrm{sc}}^{k,\ell}(\mathbb{R})$ is u in? (k is an integer.)

Problem 3.5. Consider a repulsive Coulomb-like potential V,

$$V = \frac{1}{\langle r \rangle} + S^{-1-\varepsilon}(\overline{\mathbb{R}^n}).$$

Show that $\ker_{\mathcal{S}'}(\triangle + V)$ is finite-dimensional. Hint: a coordinate change might be useful.

4. Lecture 4: Microlocalization

Recall that the *wavefront set* $\mathrm{WF}(u)$ of a distribution $u \in \mathcal{D}'(\mathbb{R})$ is a subset[1]

$$\subseteq \underbrace{\mathbb{R}^n}_{\text{base}} \times \underbrace{\partial\overline{\mathbb{R}^n}}_{\mathbb{S}^{n-1}\text{ fiber}} = \mathbb{S}^*\mathbb{R}^n \tag{4.1}$$

[1]Usually, one defines wavefront set to be a fiberwise conic subset of $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. The compactified perspective is that, instead of taking the wavefront set to consist of lines in the cotangent bundle, to only take their “endpoints” at fiber infinity.

obstructing smoothness — the wavefront set over a point $x \in \mathbb{R}^n$ in the base is a subset of the fiber $\mathbb{S}^{n-1}$ over x describing the directions where the germ of u at x fails to be smooth. A distribution is smooth if and only if it has empty wavefront set.

In this lecture, we talk about the *scattering* wavefront set. This assigns to each tempered distribution $u \in \mathcal{S}'$ a subset

$$\mathrm{WF}_{\mathrm{sc}}(u) \subset \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) \tag{4.2}$$

of the "square boundary" $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ which obstructs u being Schwartz. Over the "interior" $\mathbb{R}^n \subset \overline{\mathbb{R}^n}$, this is just the ordinary wavefront set $\mathrm{WF}(u)$. The novel thing about the sc-wavefront $\mathrm{WF}_{\mathrm{sc}}(u)$ is that it contains, in addition to the ordinary wavefront set, a set of points over spatial infinity. Namely, over $\infty\theta$, $\theta \in \mathbb{S}^{n-1}$, the sc-wavefront set contains the *frequencies* at which u, in that direction, fails to be Schwartz. So, while the ordinary wavefront set is detecting a failure to be smooth, the sc-wavefront set is also detecting failure to decay.

Obviously, we need to make this mathematically precise. We will do this in the next subsection. Afterwards, we will discuss examples.

4.1. **Basics.**

In this section, if S is a subset of $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, then we use $S^{\complement}$ to denote the complement of S within this set:

$$S^{\complement} = (\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}))\backslash S. \tag{4.3}$$

The sc-wavefront set of a tempered distribution u is defined by

$$\mathrm{WF}_{\mathrm{sc}}(u) = \bigcap_{A \in \Psi^{0,0}_{\mathrm{sc}} \text{ s.t. } Au \in \mathcal{S}} \mathrm{char}_{\mathrm{sc}}(A) = \Big(\bigcup_{A \in \Psi^{0,0}_{\mathrm{sc}} \text{ s.t. } Au \in \mathcal{S}} \mathrm{ell}_{\mathrm{sc}}(A) \Big)^{\complement}. \tag{4.4}$$

Since this is an intersection of closed sets, it is closed.

Proposition 4.1.
- *The portion of* $\mathrm{WF}_{\mathrm{sc}}(u)$ *not over base infinity is just the ordinary wavefront set* $\mathrm{WF}(u)$.
- *If u is compactly supported, then its sc-wavefront set is the same as its ordinary wavefront set.*

Proof. Exercise. □

Proposition 4.2. $u \in \mathcal{S} \iff \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing$.

Proof.
- $(\Rightarrow)$: take $A = 1$.
- $(\Leftarrow)$: if $\mathrm{WF}_{\mathrm{sc}}(u) = \varnothing$, then, for each point p in $\partial(\overline{\mathbb{R}}^n \times \overline{\mathbb{R}^n})$, there exists an $A_p \in \Psi^{0,0}_{\mathrm{sc}}$ such that $\mathrm{ell}_{\mathrm{sc}}(A_p) \ni p$ and $A_p u$ is Schwartz. Because the sets $\mathrm{ell}_{\mathrm{sc}}(A_p)$ are all open, and because $\partial(\overline{\mathbb{R}}^n \times \overline{\mathbb{R}^n})$ is compact, we can choose a finite number $J \in \mathbb{N}^+$ of points p_j such that the sets $\mathrm{ell}_{\mathrm{sc}}(A_{p_j})$ form an open cover of $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$. Now consider

$$A = \sum_{j=1}^{J} A_j^* A_j \in \Psi^{0,0}_{\mathrm{sc}}. \tag{4.5}$$

This satisfies $Au \in \mathcal{S}$. The principal symbol of A is

$$a = \sum_{j=1}^{J} |a_j|^2,$$

where the a_j's are the principal symbols of the A_j's. Because the sets $\mathrm{ell}_{\mathrm{sc}}(A_{p_j})$ form an open cover of $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$, a is totally elliptic. So, it follows that

$$Au \in \mathcal{S} \Rightarrow u \in \mathcal{S}. \tag{4.6}$$

□

A local version of this is:

Proposition 4.3. *Let $\theta \in \mathbb{S}^{n-1}$ be a direction. Then, $\mathrm{WF}_{\mathrm{sc}}(u)$ is disjoint from the fiber over $\infty\theta$ if and only if there exists some $\chi \in C^\infty(\mathbb{S}^{n-1})$ identically $= 1$ near θ and some $\psi \in C_c^\infty(\mathbb{R})$ identically $= 1$ near the origin such that the product $\chi(r^{-1}x)\psi(1/r)u$ is Schwartz.*

More generally:

Proposition 4.4. *Suppose that $A \in \Psi_{\mathrm{sc}}$ has $\mathrm{WF}'(A)$ disjoint from $\mathrm{WF}_{\mathrm{sc}}(u)$. Then Au is Schwartz.*

Proof. See Problem 4.1. □

Proposition 4.5 (Microlocality)**.** *If $A \in \Psi_{\mathrm{sc}}$ and $u \in \mathcal{S}'$, then $\mathrm{WF}_{\mathrm{sc}}(Au) \subseteq \mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{WF}'_{\mathrm{sc}}(A)$.*

Proof.
- First, we show that $\mathrm{WF}_{\mathrm{sc}}(Au) \subseteq \mathrm{WF}'_{\mathrm{sc}}(A)$, i.e. $\mathrm{WF}_{\mathrm{sc}}(Au)^{\complement} \supseteq \mathrm{WF}'_{\mathrm{sc}}(A)^{\complement}$. Suppose that $p \in \mathrm{WF}'_{\mathrm{sc}}(A)^{\complement}$. Then, $\exists B \in \Psi_{\mathrm{sc}}^{0,0}$ elliptic at p but whose operator wavefront set is disjoint from that of A. Then, $BA \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$. Thus $BAu \in \mathcal{S}$, so the elliptic set of B, including p, is contained in $\mathrm{WF}_{\mathrm{sc}}(Au)^{\complement}$.
- Second, we show that $\mathrm{WF}_{\mathrm{sc}}(Au) \subseteq \mathrm{WF}_{\mathrm{sc}}(a)$, i.e. $\mathrm{WF}_{\mathrm{sc}}(Au)^{\complement} \supseteq \mathrm{WF}_{\mathrm{sc}}(u)^{\complement}$. If $p \in \mathrm{WF}_{\mathrm{sc}}(u)^{\complement}$, then there exists a $B \in \Psi_{\mathrm{sc}}^{0,0}$ elliptic at p but with $\mathrm{WF}'_{\mathrm{sc}}(B)$ disjoint from $\mathrm{WF}_{\mathrm{sc}}(u)$. It follows that $\mathrm{WF}'_{\mathrm{sc}}(BA)$ is disjoint from $\mathrm{WF}_{\mathrm{sc}}(u)$. So, Proposition 4.4 implies that $BAu \in \mathcal{S}$. It follows that $p \in \mathrm{WF}_{\mathrm{sc}}(Au)^{\complement}$.

□

Just as differential operators do not spread supports, pseudodifferential operators do not spread singular supports or wavefront set.

The following proposition will be useful in reducing computations of sc-wavefront set to computations of ordinary wavefront set:

Proposition 4.6. *Except at the corner of the square $\partial\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}$, $\mathrm{WF}_{\mathrm{sc}}(u)$ consists of the union of the ordinary wavefront set $\mathrm{WF}(u)$ set of u and the rotation by 90° of the ordinary wavefront set of $\mathcal{F}u$.*

By rotation we mean turning the square $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ on its side.

Proof. Follows from Proposition 3.1. □

Similarly, if we want to measure u's failure to lie in some sc-Sobolev space $H_{\mathrm{sc}}^{s,\ell}$, then we consider

$$\mathrm{WF}_{\mathrm{sc}}^{s,\ell}(u) = \bigcap_{A\in\Psi_{\mathrm{sc}}^{0,0} \text{ s.t. } Au\in H_{\mathrm{sc}}^{s,\ell}} \mathrm{char}_{\mathrm{sc}}(A). \tag{4.7}$$

Proposition 4.7. $\mathrm{WF}_{\mathrm{sc}}(u) = \overline{\bigcup_{s,\ell\in\mathbb{R}} \mathrm{WF}_{\mathrm{sc}}^{s,\ell}(u)}$.

Remark 4.1. The closure here is necessary! (Why?)

The propositions above apply, mutatis mutandis, to the $\mathrm{WF}_{\mathrm{sc}}^{s,\ell}$. For example, microlocality reads:

Proposition 4.8. *If $A \in \Psi_{\mathrm{sc}}^{s,\ell}$ and $u \in \mathcal{S}'$, then $\mathrm{WF}_{\mathrm{sc}}^{s',\ell'}(Au) \subseteq \mathrm{WF}_{\mathrm{sc}}^{s'+s,\ell'+\ell}(u) \cap \mathrm{WF}'_{\mathrm{sc}}(A)$.*

4.2. **One-dimensional examples.**

Example. *Fix $\sigma > 0$. Consider $u \in \mathcal{S}'(\mathbb{R})$ defined by $u(x) = e^{i\sigma x}$. Then, $\mathrm{WF}_{\mathrm{sc}}(u)$ consists of exactly two points, one over each of the two points in $\infty\mathbb{S}^0$.*

Proof. The Fourier transform of u is a Dirac δ-function over the single point $\sigma \in \mathbb{R}$. The ordinary wavefront set of such a δ-function is the whole cosphere $\mathbb{S}^*_\sigma\mathbb{R}$ over the point where the δ-function is located. The sc-wavefront set must be the same — we cannot have any sc-wavefront set at base infinity — since the δ-function is compactly supported. So, the claim follows from Proposition 4.6. ☐

This is consistent with the intuition that $\mathrm{WF}_{\mathrm{sc}}$ is measuring which frequencies fail to decay.

Example (Wavefront set at the corner)**.** *How do we interpret sc-wavefront set at the corner*

$$\partial\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n} \subseteq \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})? \tag{4.8}$$

Let us get some intuition in the $n = 1$ case.

- *Consider the Dirac comb $u(x) = \sum_{k\in\mathbb{Z}} \delta(x-k)$. Since the singular support of u is $\mathbb{Z}$, and since the sc-wavefront set is a closed set, the sc-wavefront set of u must contain points in the corner.*

  This applies also to

  $$u(x) = \sum_{k\in\mathbb{Z}} f(x)\delta(x-k)$$

  for f Schwartz.
- *Consider $u(x) = e^{ix^2}$. This tempered distribution is not Schwartz, so it has to have sc-wavefront set* somewhere. *But it is smooth, so it has no ordinary wavefront set. Moreover, it's Fourier transform has the same form (up to rescaling) and therefore has no ordinary wavefront set either.*

  Therefore, by Proposition 4.6, the wavefront set of u must be entirely at the corner.

Intuition: sc-wavefront set at the corner corresponds to oscillations with infinite frequency which fail to decay.

4.3. **Higher-dimensional examples: it's about the cones.**

Example. *A* plane wave *has the form* $u(x) = e^{ik\cdot x}$ *for* $k \in \mathbb{R}^n$. *Claim: the sc-wavefront set of* u *consists of a single point over each point at spatial infinity.*

Proof. Same as in the one-dimensional case. □

Ignoring fiber infinity, we can think of the portion of $\partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ over $\infty\theta$, $\theta \in \mathbb{S}^{n-1}$, as consisting of one point for each plane wave.

Example. *A* spherical wave *has the form* $u(x) = e^{i\sigma\langle r\rangle}$ *for* $\sigma \in \mathbb{R}$. *The sc-wavefront set over* $\infty\theta$ *consists of a single point, that in the sc-wavefront set of the plane wave* $e^{ik\cdot x}$ *for* $k = \sigma\theta$.

Q. *Fix a plane wave* $e^{ik\cdot x}$. *At how many points do* $\mathrm{WF}_{\mathrm{sc}}(e^{ik\cdot x})$, $\mathrm{WF}_{\mathrm{sc}}(e^{i\sigma\langle r\rangle})$ *intersect?* ***A.*** 0 *if* $\sigma \neq \|k\|$, 1 *otherwise.*

Example. *Let* $n = 2$ *(for notational simplicity). Consider a "beam" of the form* $u(x) = \chi(x_2)e^{i\sigma x_1}$ *for nonnegative* $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ *not identically zero.*

Q. *What is the sc-wavefront set of* u*?* ***A.*** *The sc-wavefront set is entirely over the forward/backwards directions* $\pm\infty\mathbf{e}_1$, $\mathbf{e}_1 = (1, 0)$, *but, rather than consisting of a single point over each, it consists of (the closure of) all of the wavevectors* $k \in \mathbb{R}^2$ *of the form* $k = (\sigma, \eta)$, $\eta \in \mathbb{R}$.

Proof. First of all, u is smooth — it has no ordinary wavefront set. Its sc-wavefront set therefore sits entirely over base infinity. Some of this wavefront set could be in the corner. Let's ignore this (but see Problem 4.8). The question we are then asking is about the sc-wavefront set of u over base infinity at finite frequency. This is equivalent to asking about the ordinary wavefront set of the Fourier transform $\hat{u}$.

The Fourier transform $\hat{u}$ is $\hat{\chi}(\xi_2)\delta(\xi_1 - \sigma)$. Because χ is compactly supported, the support of $\hat{\chi}$ must be the whole real line. (This is because the Fourier transform of any C_{c}^∞ function on the real line extends to an entire, nonzero function on the complex plane, so cannot vanish on any nonempty open sets.) So, the ordinary wavefront set of $\hat{u}$ is

$$\mathrm{WF}(\hat{u}) = N^*\{\xi_1 = \sigma\}. \tag{4.9}$$

That is, it consists of the entire conormal bundle of the line $\{\xi_1 = \sigma\}$. Rotating this by $90°$, we conclude the claim. □

Moral: We cannot determine the plane waves out of which a distribution is built if we restrict attention to a rectangular prism $R = \{\|y\| < C\}$, because, if we only have access to $u|_R$, then we cannot distinguish different functions of the form $e^{ik\cdot x}A_k$ for $A_k \in C^\infty(\overline{\mathbb{R}^n})$ for $k \in \mathbb{R}^n$ with the same first component. Instead, we need access to the whole cone $\{\|y\| < Cx_1\}$.

4.4. **The microlocal elliptic parametrix construction.**

Proposition 4.9. *Suppose that* $A \in \Psi_{\mathrm{sc}}^{s,\ell}$ *is elliptic at* $p \in \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$. *Then, there exists a* $B \in \Psi_{\mathrm{sc}}^{-s,-\ell}$ *such that* $AB - 1, BA - 1$ *have* $\mathrm{WF}'_{\mathrm{sc}}$ *disjoint from* p.

This also applies with p *replaced by any closed subset.*

Proof. Analogous to construction of parametrix for totally elliptic operators. □

Microlocality says that pseudodifferential operators cannot spread wavefront set. Elliptic regularity says that pseudodifferential operators cannot kill wavefront set except where characteristic. Specifically:

Proposition 4.10. *Suppose* $A \in \Psi_{\mathrm{sc}}^{s,\ell}$. *Then,*

$$\mathrm{WF}_{\mathrm{sc}}^{s',\ell'}(Au) \supseteq \mathrm{WF}_{\mathrm{sc}}^{s',\ell'}(u) \backslash \operatorname{char}_{\mathrm{sc}}^{s,\ell}(A).$$

Proof. If $B \in \Psi_{\mathrm{sc}}^{0,0}$ and $K = I - BA$, then, from $u = BAu + Ku$, we get

$$\mathrm{WF}_{\mathrm{sc}}(u) \subseteq \mathrm{WF}_{\mathrm{sc}}(BAu) \cup \mathrm{WF}_{\mathrm{sc}}(Ku) \subseteq \mathrm{WF}_{\mathrm{sc}}(Au) \cup \mathrm{WF}'_{\mathrm{sc}}(K). \tag{4.10}$$

So,

$$\mathrm{WF}_{\mathrm{sc}}(u) \subseteq \mathrm{WF}_{\mathrm{sc}}(Au) \cup \bigcap_{B \in \Psi_{\mathrm{sc}}^{0,0}} \mathrm{WF}'_{\mathrm{sc}}(I - BA). \tag{4.11}$$

If $p \notin \operatorname{char}_{\mathrm{sc}}^{s,\ell}(A)$, then the microlocal elliptic parametrix construction says we can find some B such that the operator wavefront set of $I - BA$ does not contain p. So, p is not in the intersection above. So,

$$\bigcap_{B \in \Psi_{\mathrm{sc}}^{0,0}} \mathrm{WF}'_{\mathrm{sc}}(I - BA) \supseteq \operatorname{char}_{\mathrm{sc}}^{s,\ell}(A). \tag{4.12}$$

So we end up with $\mathrm{WF}_{\mathrm{sc}}(u) \subseteq \mathrm{WF}_{\mathrm{sc}}(Au) \cup \operatorname{char}_{\mathrm{sc}}^{s,\ell}(A)$, which is a restatement of the desired result. □

4.5. **Problems and exercises.**

Problem 4.1. Prove Proposition 4.4. Hint: you cannot use microlocality, since we used Proposition 4.4 to prove microlocality. Instead, use the microlocal elliptic parametrix construction.

Problem 4.2. Suppose that u is real-valued. Show that $\mathrm{WF}_{\mathrm{sc}}(u)$ is closed under the fiberwise antipodal map.

Problem 4.3.

- Prove $\mathrm{WF}_{\mathrm{sc}}(u + v) \subseteq \mathrm{WF}_{\mathrm{sc}}(u) \cup \mathrm{WF}_{\mathrm{sc}}(v)$.
- Let $u = \sum_{j=1}^{J} e^{ik_j \cdot x}$ for some distinct $k_j \in \mathbb{R}^n$. What is $\mathrm{WF}_{\mathrm{sc}}(u)$?

Problem 4.4. Let $p(x)$ denote a nonzero polynomial of $x \in \mathbb{R}^n$. What can the sc-wavefront set of p be? Can you guess the answer before doing any work?

Problem 4.5. Consider $u(x) = e^{ix^3} \in \mathcal{S}'(\mathbb{R})$. What is the sc-wavefront set of this u? Can you guess the answer before doing any work?

Problem 4.6. Consider the Bessel function $J_\nu(x)$, $\nu \in \mathbb{R}$. What is the sc-wavefront set of $u(x) = 1_{x>0}\chi(1/x)J_\nu(x)$?

Hint: use Bessel's ODE, $x^2u''(x) + xu'(x) + (x^2 - \nu^2)u(x) = 0$.

Problem 4.7. Consider the Airy function $A(x)$. What is the sc-wavefront set of A?

Hint: use Airy's ODE $A''(x) = xA(x)$.

Problem 4.8. Consider the beam $u(x) = \chi(x_2)e^{i\sigma x_1}$ on $\mathbb{R}^2$. Show that the portion of its sc-wavefront set at the corner is the closure of the sc-wavefront set at finite frequency.

Problem 4.9. Repeat a nonempty subset of the problems above for the Sobolev wavefront sets $\mathrm{WF}^{m,s}_{\mathrm{sc}}$.

A *perturbed plane wave* is a function $u \in C^\infty(\mathbb{R}^n)$ of the form $u(x) = e^{ik\cdot x}A(x)$ for some nonzero $k \in \mathbb{R}^n$ and $A \in C^\infty(\overline{\mathbb{R}^n})$ such that $A|_{\infty\mathbb{S}^{n-1}} = 1$. The function $e^{ik\cdot x}$ is an exact plane wave, and $e^{ik\cdot x}(A-1)$ is the perturbation. Perturbed plane waves are one of the basic objects of scattering theory. A natural question to ask is:

Q. how large does a subset S of $\mathbb{R}^n$ need to be for k to be uniquely determined by $u|_S$?

In other words, for what S can we determine the wavevector k from only looking at u within S?

Of course, no bounded subset of $\mathbb{R}^n$ will do, because we can prescribe A arbitrarily in bounded subsets. We restrict attention to the region

$$S[C,j] = \{x \in \mathbb{R}^n : x_1 > 0, \|y\| \le Cx_1^j\} \tag{4.13}$$

where $x = (x_1, y)$, for $C > 0$, $j \ge 0$.

Problem 4.10. For which regions $S = S[C,j]$ is k uniquely determined by $u|_S$? Say what you can.

5. Microlocal propagation estimates I

5.1. **Null bicharacteristics and microlocal propagation.**

Consider $P \in \Psi^{m,l}_{\mathrm{sc}}(\mathbb{R}^n)$ which is not elliptic. The failure of ellipticity could be at frequency infinity, at spatial infinity, or both. For example, the Helmholtz operator $\Delta - \lambda^2$, where $\lambda > 0$, is elliptic at frequency infinity but not at spatial infinity, while the Klein Gordon operator $D_t^2 - \Delta - m^2$, $m > 0$, is non-elliptic both at frequency infinity and at spatial (or more precisely spacetime) infinity. We'll assume that P has a real, classical principal symbol $p \in S^{m,l}_{\mathrm{sc,cl}}$.

Recall some basic symplectic geometry: the vector field H_p is given by the formula

$$H_p = \sum_j \Big(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big). \tag{5.1}$$

Although we write this using the usual Cartesian coordinates on $\mathbb{R}^n$, this is actually invariant under coordinate changes: if $(x_1, \dots x_n)$ are *any* local coordinates and $(\xi_1, \dots, \xi_n)$ are the dual coordinates induced by $(x_1, \dots x_n)$ on the fibres of the cotangent bundle, then H_p takes the form (5.1) in these coordinates.

If P happens to have order $(1,1)$, then the Hamilton vector field is a smooth vector field on $\overline{T^*\mathbb{R}^n}$, tangent to the boundary, and hence restricts to a smooth vector field on $\partial\overline{T^*\mathbb{R}^n}$. This is straightforward to see from (5.1), which we write as follows:

$$H_p = \sum_j \Big(\langle x\rangle^{-1}\frac{\partial p}{\partial \xi_j}(\langle x\rangle\frac{\partial}{\partial x_j}) - \langle \xi\rangle^{-1}\frac{\partial p}{\partial x_j}(\langle \xi\rangle\frac{\partial}{\partial \xi_j})\Big). \tag{5.2}$$

As we have seen before, the vector fields $\langle x\rangle\partial_{x_j}$ and $\langle\xi\rangle\partial_{\xi_j}$ are smooth on $\overline{T^*\mathbb{R}^n}$ and tangent to the boundary, while the coefficients in (5.2) are classical of order $(0,0)$ and hence smooth on $\overline{T^*\mathbb{R}^n}$. For operators of general order (m,l), it is convenient to rescale the Hamilton vector field as follows:

$$^{\mathrm{sc}}H_p^{m,l} = \langle x\rangle^{-l+1}\langle\xi\rangle^{-m+1}H_p. \tag{5.3}$$

Then ${}^{\mathrm{sc}}H_p^{m,l}$ is again smooth on $\overline{T^*\mathbb{R}^n}$ and tangent to the boundary. Moreover, the flow of ${}^{\mathrm{sc}}H_p^{m,l}$ exists for all 'time'. An easy calculation (see exercises) shows that this rescaled vector field is tangent to $\mathsf{p} = 0$, where $\mathsf{p} = \langle x\rangle^{-l}\langle\xi\rangle^{-m}p$ is a smooth function on $\overline{T^*\mathbb{R}^n}$ vanishing at $\mathrm{Char}(P)$. It follows that if one point of an integral curve of H_p is contained in $\mathrm{Char}(P)$, then the whole integral curve is contained in $\mathrm{Char}(P)$. Such integral curves play an important role in microlocal analysis.

Definition 5.1. *A (null-)bicharacteristic of P is an integral curve of ${}^{\mathrm{sc}}H_p^{m,l}$ contained in* $\mathrm{Char}(P) = \{\mathsf{p} = 0\} \cap \partial\overline{T^*\mathbb{R}^n}$.

Simple example: take $P = D_{x_1} \in \Psi^{1,0}$. Then $p = \xi_1$ and $H_p = \partial_{x_1}$. The regularized Hamilton vector field is ${}^{\mathrm{sc}}H_p^{1,0}$ is $\langle x\rangle\partial_{x_1}$ and is tangent to $\partial\overline{T^*\mathbb{R}^n}$. Note that if we are only interested in frequency infinity, then over a bounded region in x-space there is no need for the regularization.

5.2. **Propagation theorems.** A major theorem (or really a meta-theorem, with many versions in different contexts) in microlocal analysis is called 'Propagation of Singularities', or perhaps more accurately 'Propagation of Regularity'. Our first version of this theorem is:

Theorem 5.1. *Suppose that $P \in \Psi_{\mathrm{sc}}^{m,l}(\mathbb{R}^n)$ and admits a real, classical principal symbol p. Suppose that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $Pu \in \mathcal{S}(\mathbb{R}^n)$. Then*

- $\mathrm{WF}_{\mathrm{sc}}(u) \subset \mathrm{Char}(P)$*, and*
- $\mathrm{WF}_{\mathrm{sc}}(u)$ *is a union of bicharacteristics of* ${}^{\mathrm{sc}}H_p^{m,l}$.

That is, if $q, q' \in \mathrm{Char}(P)$ are on the same bicharacteristic of P, then $q \in \mathrm{WF}_{\mathrm{sc}}(u)$ iff $q' \in \mathrm{WF}_{\mathrm{sc}}(u)$. Equivalently, $q \notin \mathrm{WF}_{\mathrm{sc}}(u)$ iff $q' \notin \mathrm{WF}_{\mathrm{sc}}(u)$, i.e. we have 'propagation of regularity' along bicharacteristics.

We can also give a more quantitative version of this, in which we look at wavefront set relative to $H^{s,k}$:

Theorem 5.2. *Suppose that $P \in \Psi_{\mathrm{sc}}^{m,l}(\mathbb{R}^n)$ and admits a real, classical principal symbol p. Suppose that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $Pu \in H^{s,k}$. Then*

- *u is in $H^{s+m,k+l}$ microlocally on* $\mathrm{Ell}(P)$*, i.e.*
$$\mathrm{WF}_{\mathrm{sc}}^{s+m,k+l}(u) \cap \mathrm{Ell}(P) = \emptyset;$$
- $\mathrm{WF}_{\mathrm{sc}}^{s+m-1,k+l-1}(u) \subset \mathrm{Char}(P)$ *is a union of bicharacteristics of* ${}^{\mathrm{sc}}H_p^{m,l}$.

Remark 5.1. Notice the discrepancy of the two orders in the two statements of Theorem 5.2. On the elliptic set, we gain regularity of order (m, l), the same as the order of P, where 'gain' refers to the orders of the Sobolev space containing u compared to the orders of the Sobolev space containing Pu. On the characteristic variety, there is no 'automatic' gain as in the elliptic case, but there is a 'conditional' gain, i.e. if one has regularity of a certain order of Pu and regularity of u at some point along a bicharacteristic, then one has regularity along the whole bicharacteristic. However, the 'non-elliptic' gain here is only $(m - 1, l - 1)$, one less in each exponent relative to the elliptic gain. This is always the case and is explained by the method of proof below. More on this later in Lecture 6.

We will prove a *microlocal propagation estimate* that implies Theorem 5.2 and is a strictly stronger result, as it is 'fully microlocal': instead of the global assumption that $Pu \in H^{k,r}$, we only assume this in some microlocal region. To set up this theorem, we suppose as before that $P \in \Psi_{\mathrm{sc}}^{m,l}$ has real, classical principal symbol, and $\gamma : [0, s_0] \to \mathrm{Char}(P)$ is a nontrivial bicharacteristic, i.e. nonconstant (equivalent to assuming that ${}^{\mathrm{sc}}H_p^{m,l}$ does not vanish at $\gamma(0)$). Let U_0 and U be open subsets of $\partial\overline{T^*\mathbb{R}^n}$ such that U_0 contains $\gamma(0)$ and U contains $\gamma([0, s_0])$.

Theorem 5.3. *There exist operators $B, E, G \in \Psi_{\mathrm{sc}}^{0,0}$ such that*

$$\begin{gathered}\gamma([0, s_0]) \subset \mathrm{Ell}(B) \subset \mathrm{WF}'(B) \subset \mathrm{Ell}(G) \subset \mathrm{WF}'(G) \subset U, \\ \gamma(0) \subset \mathrm{Ell}(E) \subset \mathrm{WF}'(E) \subset U_0,\end{gathered} \tag{5.4}$$

such that for all s, k, N there exists $C > 0$ such that for all $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\|Bu\|_{H^{s,k}} \leq C\Big(\|GPu\|_{H^{s-m+1,k-l+1}} + \|Eu\|_{H^{s,k}} + \|u\|_{H^{-N,-N}}\Big). \tag{5.5}$$

This inequality holds in the strong sense that if the RHS is finite, then so is the LHS, and the inequality holds. In particular it means that if $u \in H^{-N,-N}$, Pu is in $H^{s-m+1,k-l+1}$ microlocally in U, and u is in $H^{s,k}$ microlocally in U_0, then u is in $H^{s,k}$ microlocally on $\mathrm{Ell}(B)$*, in particular on $\gamma([0, s_0])$.*

Remark 5.2. Note that this estimate is 'fully microlocal' in the sense that the assumptions both on u and Pu are microlocalized near the region of interest, which is the bicharacteristic segment $\gamma([0, s_0])$. The only exception is the assumption that $u \in H^{-N,-N}$ but this is really no assumption at all, since every tempered distribution is in $H^{-N,-N}$ for sufficiently large N.

We will prove this result next lecture.

5.3. **A simple example.** In the remainder of this lecture we will present a proof in a simple model situation. Here will be working at frequency infinity, over a bounded region of $\mathbb{R}^2$. Let $P = D_{x_1} \in \Psi_{\mathrm{sc}}^{1,0}(\mathbb{R}^2)$, and we suppose that u is a tempered distribution such that $Pu = f \in L^2(\mathbb{R}^2)$. We will assume that

$$u \in L^2([-2, 0]_{x_1} \times [-2, 2]_{x_2}).$$

The problem is to prove that u is L^2 on a rectangle that is larger in the x_1 direction, although (for convenience) a bit smaller in the L^2 direction: show that

$$u \in L^2([-2, 2]_{x_1} \times [-1, 1]_{x_2}).$$

That is, regularity of u 'propagates' in the x_1-direction. It will be enough, in view of the assumption on u, to show it is L^2 on $L^2([-1, 2]_{x_1} \times [-1, 1]_{x_2})$.

We choose a cutoff function $\chi_1(x_1)$ that is identically 1 on $[-1, 2]$, supported on $[-2, 3]$, and monotone nondecreasing between $[-2, -1]$, monotone nonincreasing on $[2, 3]$. We also choose a cutoff $\chi_2(x_2)$ that is identically 1 on $[-1, 1]$, supported on $[-2, 2]$, and monotone nondecreasing between $[-2, -1]$, monotone nonincreasing on $[1, 2]$. We define

$$a(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)e^{-x_1}.$$

We compute

$$\begin{aligned} i[P, a(x_1, x_2)] &= \partial_{x_1} a(x_1, x_2) \\ &= -b(x_1, x_2) + e(x_1, x_2) \text{ where} \\ b(x_1, x_2) &= a(x_1, x_2) - \chi_1'(x_1) 1_{x_1 \geq 0} e^{-x_1} \chi_2(x_2) \geq 0, \\ e(x_1, x_2) &= \chi_1'(x_1) 1_{x_1 \leq 0} e^{-x_1} \chi_2(x_2) \geq 0. \end{aligned} \tag{5.6}$$

Then we compute (for suitably regular u, say $u \in C^1(\mathbb{R}^2)$) the commutator

$$i\big\langle i[P, a(x_1, x_2)]u, u\big\rangle_{L^2(\mathbb{R}^2)} \tag{5.7}$$

in two different ways.

First, we unwrap the commutator, obtaining

$$\begin{aligned} (5.7) &= i\big\langle a(x_1, x_2)u, Pu\big\rangle - i\big\langle Pu, a(x_1, x_2)u\big\rangle \\ &= 2\operatorname{Im}\big\langle Pu, a(x_1, x_2)u\big\rangle = 2\operatorname{Im}\big\langle f, a(x_1, x_2)u\big\rangle. \end{aligned} \tag{5.8}$$

Second, we use (5.6) to write

$$(5.7) = -\langle bu, u\rangle + \langle eu, u\rangle. \tag{5.9}$$

Using the support property of e, we put these two identities together and use Cauchy-Schwarz on (5.7) to find that

$$\langle bu, u\rangle \leq \langle eu, u\rangle + \frac{1}{\epsilon}\|f\|_{L^2}^2 + \epsilon\|au\|_{L^2}^2$$

for any $\epsilon > 0$. Here, the $\langle eu, u\rangle$ term can be controlled by $C'\|u\|^2_{L^2([-2,1]_{x_1}\times[-2,2]_{x_2})}$, but we do not make this step just yet.

Notice that $0 \leq a \leq 9$. So $a^2 \leq 9a \leq 9b$. Choosing $\epsilon = 1/18$, we find

$$\langle bu, u\rangle \leq C'\langle eu, u\rangle + 18\|f\|_{L^2}^2 + \frac{1}{2}\langle bu, u\rangle.$$

We now absorb the $\langle bu, u\rangle$ term on the RHS and obtain, after multiplying by 2,

$$\langle bu, u\rangle \leq 2C'\langle eu, u\rangle + 36\|f\|_{L^2}^2.$$

Unfortunately, this argument does *not* prove that u is L^2 on $[-1, 2] \times [-1, 1]$ as it requires that $\langle bu, u\rangle < \infty$ (for the absorption step), which is a stronger assumption! So it might seem that what we've done so far is completely pointless. This is not the case however, as we have proved a *quantitative* estimate that can be combined with an approximation argument to achieve our aim. Quantitative estimates are very powerful as we shall see shortly.

To do this, we notice that we can assume without loss of generality that u is compactly supported, and any compactly supported tempered distribution is in H^{-N}, for sufficiently large N. Fixing such an N, we let

$$u_r = (1 + r\Delta)^{-M}u := T_r u, \quad M \geq N/2, r > 0,$$

and consider the limit as $r \to 0$. Intuitively, as $r \to 0$, T_r tends to the identity operator. In fact this is rigorously true in the topology of $\Psi_{\mathrm{sc}}^{\eta,\eta}$ for any $\eta > 0$; moreover, T_r is uniformly bounded in $\Psi_{\mathrm{sc}}^{0,0}$ as $r \to 0$ and converges strongly to the identity operator in $\mathcal{B}(L^2, L^2)$ (exercise). We have

$$u_r \in L^2 \text{ for all } r > 0, \text{ and } Pu = T_r f = f_r$$

where for the second identity we used the fact that P and T_r commute (both are Fourier multipliers). Our previous estimate applies to u_r for each $r > 0$, and we obtain

$$\langle bu_r, u_r\rangle \le 2C'\langle eu_r, u_r\rangle + 36\|f_r\|_{L^2}^2. \tag{5.10}$$

Now we take the limit as $r \to 0$. Since $f \in L^2$, we have $f_r \to f$ in L^2 and hence

$$\|f_r\|_{L^2}^2 \to \|f\|_{L^2}^2.$$

Next we consider the limit of the $\langle eu_r, u_r\rangle$ term. This is slightly more complicated as u, unlike f, is not globally in L^2. We write this term by choosing a new cutoff function ψ, so that $\psi \equiv 1$ on the support of e, and ψ is supported in $[-2, 1]_{x_1} \times [-2, 2]_{x_2}$. We have

$$\langle eT_r u, T_r u\rangle = \langle eT_r\psi u, T_r\psi u\rangle + \langle eT_r(1-\psi)u, T_r\psi u\rangle + \langle T_r u, eT_r(1-\psi)u\rangle. \tag{5.11}$$

The first term on the RHS converges to $\langle eu, u\rangle$ since $\psi u \in L^2$, $T_r \to \mathrm{Id}$ strongly, and $\psi \equiv 1$ on the support of e. We would like to show that the other two terms tend to zero.

Consider the operator $eT_r(1-\psi)$. Since $e(1-\psi) = 0$ (as the supports of e and $1-\psi$ are disjoint) this operator is equal to $e(T_r - \mathrm{Id})(1-\psi)$. We recall that $T_r - \mathrm{Id}$ converges to zero in the topology of $\Psi_{\mathrm{sc}}^{\eta,\eta}$ for all $\eta > 0$. However, we can say much more by writing the operator $eT_r(1-\psi)$ as the expansion of this composition to K terms, plus a remainder term. The expansion vanishes identically, and the remainder (which is $eT_r(1-\psi)$ itself) tends to zero in $\Psi_{\mathrm{sc}}^{-K+\eta,-K+\eta}$ for all $\eta > 0$. From this we see that $eT_r(1-\psi)u$ converges to zero in every Sobolev space, showing that the second and third terms on the RHS tend to zero.

Using (5.10) and the reasoning above, we have

$$\limsup_{r\to 0}\langle bu_r, u_r\rangle \le 2C'\langle eu, u\rangle + 36\|f\|_{L^2}^2. \tag{5.12}$$

Now we would like to distribute $\sqrt{b}$ on both sides of the inner product on the LHS. This can be done either by choosing b to be a square of a smooth function from the outset, or we can give up a little, and choose $\chi \in C_c^\infty(\mathbb{R}^2)$ such that $0 \le \chi^2 \le b$ and $\chi \equiv 1$ on $[-2, 1] \times [-1, 1]$. Then we have from (5.13) that

$$\limsup_{r\to 0}\|\chi T_r u\|_{L^2}^2 \le 2C'\langle eu, u\rangle + 36\|f\|_{L^2}^2. \tag{5.13}$$

The uniform boundedness of $\chi T_r u$ in L^2 means that there is a weakly convergent subsequence, (weakly) converging to $v \in L^2$, say. A fundamental property of weak limits is that $\|v\|_{L^2}^2$ also satisfies the inequality (5.13). On the other hand, we have $\chi T_r u \to \chi u$ in a weaker topology, that of $H^{-N-\eta,-N-\eta}$, using the convergence of $T_r \to \mathrm{Id}$ in $\Psi_{\mathrm{sc}}^{\eta,\eta}$ for all $\eta > 0$. Uniqueness of limits shows that $v = \chi u$. We deduce that

$$\|\chi u\|_{L^2}^2 \le 2C'\langle eu, u\rangle + 36\|f\|_{L^2}^2 \le 2C'\|u\|_{L^2([-2,1]_{x_1}\times[-2,2]_{x_2})}^2 + 36\|f\|_{L^2}^2, \tag{5.14}$$

and since $\chi = 1$ on $[-1, 2] \times [-1, 1]$, we have achieved our goal.

5.4. **Exercises.**

(1) Check that the rescaled Hamilton vector field ${}^{\mathrm{sc}}H_p^{m,l}$ of $P = \mathrm{Op}(p)$ is tangent to $\mathsf{p} = 0$, using the notation in the lecture notes above.
(2) Show that $T_r = (1 + r\Delta)^{-M}$ is a uniformly bounded family of operators in $\Psi_{\mathrm{sc}}^{0,0}$, for any $M > 0$.
(3) Suppose that u is a tempered distribution in $\mathbb{R}^2$ satisfying the PDE
$$D_{x_1} u = \delta + f,$$
where δ is the two-dimensional delta distribution, supported at the origin in $\mathbb{R}^2$, and f is an L^2 function of $(x_1, x_2) \in \mathbb{R}^2$ supported where $x_1 > 0$. You are given that u is C^∞ for $x_1 < 0$. Use the result of Theorem 5.2 to answer the following questions:
(i) Suppose that $s < -1$. Show that u is 'locally' in $H^s(\mathbb{R}^2)$, in the sense that for any $\phi \in C_c^\infty(\mathbb{R}^2)$, we have $\phi u \in H^s(\mathbb{R}^2)$.
(ii) Suppose that $-1 < s < 0$. Give an optimal bound on $\mathrm{WF}^s(u)$. (Hint: it might help to first consider the case that $f = 0$.)
(iii) Suppose that $s > 0$. Give an optimal bound on $\mathrm{WF}^s(u)$, expressed in terms of $\mathrm{WF}^s(f)$ (and taking account of the δ term as well).

6. Microlocal propagation estimates II

In this section we prove Theorem 5.3 for a general operator P. However, we shall assume that $P \in \Psi_{\mathrm{sc}}^{1,1}$ which simplifies some of the numerology. There is no real loss of generality, as solving an equation $Pu = f$ is equivalent to solving $TPu = Tf$ where T is an invertible operator. We can always choose an invertible scattering pseudodifferential operator T with real, classical symbol, such that $TP \in \Psi_{\mathrm{sc}}^{1,1}$ (see below) and this reduces us to this case.

For such a P, with real, classical principal symbol, we want to construct $B, E, G \in \Psi_{\mathrm{sc}}^{0,0}$ as in the theorem, so that the estimate
$$\|Bu\|_{H^{s,k}} \le C\Big(\|GPu\|_{H^{s,k}} + \|Eu\|_{H^{s,k}} + \|u\|_{H^{-N,-N}} \Big) \tag{6.1}$$
holds. The nontrivial case of (6.1) is that u is in $H^{-N,-N}$, that Pu is in $H^{s,k}$ microlocally in U, and that u is in $H^{s,k}$ microlocally in U_0, otherwise the RHS is infinite (or can be made infinite by choosing E, G carefully) and the result is vacuously true. So we assume this for the remainder of the proof.

6.1. **Proof of Theorem 5.3, Step 1.** We construct $A \in \Psi_{\mathrm{sc}}^{2s,2k}$ with real symbol a, with $A = A^*$ such that
$$i(P^*A - AP) = -\tilde{B}^*\tilde{B} - (\Lambda^{-s,-r}A)^*\Lambda^{-s,-r}A + E' + R, \tag{6.2}$$
where

- $\Lambda^{s,k} = \langle x \rangle^k \langle D \rangle^s$ is an invertible elliptic operator, with real, classical symbol, of order (s, k). Note that $\langle D \rangle^s$ is an abbreviation for $\mathrm{Op}(\langle \xi \rangle^s)$.
- $\tilde{B} = \Lambda^{s,k} B$, where B is as in (6.1).
- $E' \in \Psi_{\mathrm{sc}}^{2s,2k}$ and $\mathrm{WF}'(E') \subset \mathrm{Ell}(E)$.
- $R \in \Psi_{\mathrm{sc}}^{2s-1,2k-1}$, $\mathrm{WF}'(R) \subset \mathrm{Ell}(G)$.

- G is microlocally equal to the identity on $\mathrm{WF}'(A)$.

Notice that in (6.4), the LHS has order $(2s, 2k)$ although the individual terms P^*A and AP have higher order. In fact, we can write this as

$$i[P, A] + i(P^* - P)A,$$

and $[P, A]$ has order $(2s, 2k)$ since commutators drop order by $(1, 1)$ relative to the sum of the orders of the individual operators; while $(P^* - P)$ has order $(0, 0)$ since $\sigma^{1,1}_{\mathrm{pr}}(P) = \overline{\sigma^{1,1}_{\mathrm{pr}}(P)} = \sigma^{1,1}_{\mathrm{pr}}(P^*)$. On the RHS side, the first three terms have order $(2s, 2k)$, while the last term has order $(2s - 1, 2k - 1)$. Thus, if we view the R term as a remainder term of lesser interest (we only care about it microsupport, as in the fourth bullet point above), then the operator equation (6.4) can be arranged by making it hold at a principal symbol level.

The reason for being interested in the quantity $i(P^*A - AP)$ is that

$$\big\langle i(P^*A - AP)u, u\big\rangle = 2\,\mathrm{Im}\langle Pu, Au\rangle.$$

Given the identity (6.4), and a sufficiently nice u (meaning it is in a Sobolev space with sufficiently large orders), we can make the following calculation:

$$2\,\mathrm{Im}\langle Pu, Au\rangle = -\|\tilde{B}u\|^2_{L^2} - \|\Lambda^{-s,-k}Au\|^2_{L^2} + \langle E'u, u\rangle + \langle Ru, u\rangle. \tag{6.3}$$

We can microlocalize the Pu term by using the fact that $G = \mathrm{Id}$ microlocally on $\mathrm{WF}'(A)$. Therefore, by redefining R by the addition of a residual operator (which we do not indicate in notation), we obtain from (6.4) the variant

$$i((GP)^*A - AGP) = -\tilde{B}^*\tilde{B} - (\Lambda^{-s,-k}A)^*\Lambda^{-s,-k}A + E' + R, \tag{6.4}$$

and this leads to (with the same redefinition of R)

$$2\,\mathrm{Im}\langle GPu, Au\rangle = -\|\tilde{B}u\|^2_{L^2} - \|\Lambda^{-s,-k}Au\|^2_{L^2} + \langle E'u, u\rangle + \langle Ru, u\rangle. \tag{6.5}$$

in place of (6.3). We estimate the LHS as follows: we write $Au = (\Lambda^{-s,-k})^{-1}\Lambda^{-s,-k}Au$ and move the left factor to the other side of the inner product, noting that the adjoint of $(\Lambda^{-s,-k})^{-1}$ is $\Lambda^{s,k}$. We then apply Cauchy-Schwarz to the inner product. We obtain

$$\|\tilde{B}u\|^2_{L^2} + \|\Lambda^{-s,-k}Au\|^2_{L^2} \le \|\Lambda^{s,k}GPu\|_{L^2}\|\Lambda^{-s,-k}Au\|_{L^2} + \langle E'u, u\rangle + \langle Ru, u\rangle. \tag{6.6}$$

We now apply the inequality $2ab \le a^2 + b^2$ to the first term on the RHS. We also apply an elliptic estimate to the $\langle E'u, u\rangle$ term, noting that $(\Lambda^{s,r}E)^*\Lambda^{s,r}E$ is elliptic on $\mathrm{WF}'(E')$ (see exercises at the end of the lecture). We can do the same with the R term. In fact, we choose a $B' \in \Psi^{0,0}_{\mathrm{sc}}$ such that $\mathrm{WF}'(R) \subset \mathrm{Ell}(B') \subset WF'(B') \subset \mathrm{Ell}(G)$, and estimate it in the same way.

$$\begin{aligned}\|\Lambda^{s,k}Bu\|^2_{L^2} + \|\Lambda^{-s,-r}Au\|^2_{L^2} &\le \|\Lambda^{s,r}GPu\|^2_{L^2} + \|\Lambda^{-s,-r}Au\|^2_{L^2} \\ &\quad + C\|\Lambda^{s,r}Eu\|^2_{L^2} + C\|\Lambda^{s-1/2,r-1/2}B'u\|^2_{L^2} + C\|u\|^2_{H^{-N,-N}},\end{aligned} \tag{6.7}$$

where the final term arises from the remainder term in the elliptic estimates. The $\|\Lambda^{-s,-r}Au\|^2_{L^2}$ terms cancel. Now eliminating the $\Lambda^{\bullet,\bullet}$ factors and writing norms in terms of weighted Sobolev norms, we have obtained

$$\|Bu\|^2_{H^{s,r}} \le C\Big(\|GPu\|^2_{H^{s,r}} + \|Eu\|^2_{H^{s,r}} + \|B'u\|^2_{H^{s-1/2,r-1/2}} + \|u\|^2_{H^{-N,-N}}\Big), \tag{6.8}$$

which is almost the estimate we are aiming for: the only discrepancy is the term $C\|B'u\|^2_{H^{s-1/2,r-1/2}}$ on the RHS. Notice that this term is just like the Bu term on the LHS that we are estimating, but it has orders lower by $(1/2, 1/2)$. We can thus proceed by induction, estimating the B' term as above, up to an additional $B''u$ term that would be measured in the Sobolev space $H^{s-1,r-1}$, and so on. Each time, we need to enlarge the microlocal support of $B', B'', \ldots$ relative to the operator before (and also enlarge the microlocal support of the E' term correspondingly), but we can do this while always remaining in the region where G is microlocally equal to the identity. After a finite number of steps, the order of the extra term is reduced to $(-N, -N)$ and then this term can be absorbed in the $\|u\|^2_{H^{-N,-N}}$ term, at which point the argument is terminated and the estimate is proved.

To complete the proof, we need to accomplish two more steps: arrange the operator identity (6.4), and eliminate the assumption that u is in a sufficiently nice Sobolev space.

Remark 6.1. Returning to Remark 5.1, the reason that there is a loss of 1 in both orders relative to the elliptic gain is that, in the propagation proof, the estimate arises from the ellipticity of B, which comes from the positivity (actually negativity, as we presented it!) of the commutator $i[P, A]$. On the other hand, in the elliptic estimate the ellipticity is directly from the ellipticity of P. Since the commutator drops order by $(1, 1)$ relative to the composition of the two operators, this causes the propagation estimate to be weaker, i.e. we need to measure Pu by a stronger norm to deduce a fixed Sobolev norm of Bu.

6.2. **Proof of Theorem 5.3, Step 2.** Constructing the operators. We do this first in the case that $(s, k) = (0, 0)$. The general case is hardly more complicated.

We need to arrange (6.4). Since we are not much interested in the properties of R (other than its microlocal support, which is anyway bounded above by the union of the microlocal supports of the other operators), to have (6.4) it is enough to ensure it holds at a principal symbol level. Let $p_1 = i\sigma_L(P^* - P) \in S^{0,0}_{sca}$; this is a classical symbol, and real since $i(P^* - P)$ is symmetric. At the principal symbol level, (6.4) reads

$$H_p(a) + p_1 a = -b^2 - a^2 + e'. \tag{6.9}$$

Using ODE theory, assuming that γ is not contained in the corner of $\overline{T^*\mathbb{R}^n}$, there are local coordinates (z_1, z'), $z' \in \mathbb{R}^{2n-2}$, on $\partial\overline{T^*\mathbb{R}^n}$ near $\gamma([0, s_0])$ so that $\gamma(0) = (0, 0)$ and $H_p = \partial_{z_1}$ in this coordinate system, i.e. bicharacteristics are given by z' =constant. A small, mostly notational, modification of this construction serves to treat the case that γ is contained in the corner of $\overline{T^*\mathbb{R}^n}$; we do not pursue this further here.

Choose $\epsilon > 0$ sufficiently small so that, in the coordinates (z_1, z'),

$$[-2\epsilon, s_0 + 2\epsilon] \times \{|z'| \le 2\epsilon\} \subset U \subset WF^{0,0}(Pu)^{\complement},$$
$$[-2\epsilon, 2\epsilon] \times \{|z'| \le 2\epsilon\} \subset U_0 \subset WF^{0,0}(u)^{\complement}.$$

We now choose several cutoff functions. Let $\psi(z') \in C_c^\infty(\mathbb{R}^{2n-2})$ be such that $\psi(z') = 1$ for $|z'| \le \epsilon$ and $\psi(z') = 0$ if $|z'| \ge 2\epsilon$. Then, in the z_1 direction we choose turn-on and turn-off functions. The turn-on function is $\chi_1(z_1)$ which is smooth, monotone, equal to 0 for $z_1 \le -\epsilon$ and 1 for $z_1 \ge \epsilon$. The turn-off function we specify a bit more explicitly, as we need to take square-roots and show smoothness of these. We let χ_0 be the smooth function defined by

$$\chi_0(t) = \begin{cases} 0, t \leq 0 \\ e^{-\digamma/t}, t \geq 0 \end{cases} \tag{6.10}$$

where $\digamma > 0$ will be chosen sufficiently large. Notice that

$$\chi_0'(x) = \digamma \chi_0(x)/x^2. \tag{6.11}$$

The turn-off function is then defined by

$$\chi(z_1) = \chi_0(s_0 + \epsilon - z_1) \Longrightarrow \chi'(z_1) = -\frac{\digamma}{(s_0 + \epsilon - z_1)^2}\chi(z_1). \tag{6.12}$$

We now define

$$a(z, z') = \chi_1(z_1)^2 \chi(z_1) \psi(z')^2. \tag{6.13}$$

We compute

$$\begin{gathered} H_p a + p_1 a = \chi'(z_1)\chi_1(z_1)^2\psi(z')^2 \\ +p_1(z, z')\chi(z_1)\chi_1(z_1)^2\psi(z')^2 \\ +2\chi_1(z_1)\chi_1'(z_1)\chi(z_1)\psi(z')^2 \end{gathered} \tag{6.14}$$

where the sum of the first two lines is nonpositive (for $\digamma$ sufficiently large) and the third line is nonnegative. We will define b so that the first two lines are $-b^2 - a^2$, and the third line is e'. Using (6.12), this requires that

$$b^2 = \Big(\frac{\digamma}{(s_0 + \epsilon - z_1)^2} + p_1\Big)\chi(z_1)\chi_1(z_1)^2\psi(z')^2 - \chi(z_1)^2\chi_1(z_1)^4\psi(z')^4. \tag{6.15}$$

Noting that $\sqrt{\chi_1}$ is a smooth function, we have

$$b = \sqrt{\chi(z_1)}\chi_1(z_1)\psi(z')\sqrt{\frac{\digamma}{(s_0 + \epsilon - z_1)^2} - p_1 - \chi(z_1)\chi_1(z_1)^2\psi(z')^2} \tag{6.16}$$

Clearly, for $\digamma$ sufficiently large, the argument of the square root is bounded away from zero on the support of $\chi_1(z_1)\chi(z_1)\psi(z')^2$ and therefore b is smooth, and can be extended to $\overline{T^*\mathbb{R}^n}$ as a symbol of order $(0, 0)$.

As mentioned above, we define e' to be the third line of (6.14):

$$e' = 2\chi_1(z_1)\chi_1'(z_1)\chi(z_1)\psi(z')^2. \tag{6.17}$$

We choose g, the principal symbol of G, to be a smooth function equal to 1 on the support of a (hence also on the support of b and e) and supported in U.

We then extend the smooth functions a, b, e', g into the interior and quantize to obtain operators

$$A = \frac{\mathrm{Op}_L(a) + \mathrm{Op}_L(a)^*}{2}, \quad B = \mathrm{Op}_L(b), \quad E' = \mathrm{Op}_L(e'), \quad G = \mathrm{Op}_L(g).$$

so that (6.4) is satisfied, noting that this makes $R \in \Psi_{\mathrm{sc}}^{-1,-1}$. We must extend g into the interior so that it is identically 1 in a neighbourhood (in the ambient space, i.e. not just at $\partial\overline{T^*\mathbb{R}^n}$) of $\mathrm{WF}'(A)$. We also note that $\mathrm{WF}'_{\mathrm{sc}}(R)$ is bounded by the union of $\mathrm{WF}'_{\mathrm{sc}}(A), \mathrm{WF}_{\mathrm{sc}}(B)$ and $\mathrm{WF}_{\mathrm{sc}}(E')$. Thus (6.4), and the conditions listed below that equation, are all satisfied.

In the case of general (s,k) we redefine a to be

$$a = \langle \xi \rangle^{2s} \langle x \rangle^{2k} \chi_1(z_1)^2 \chi(z_1) \psi(z')^2$$

and proceed as before. In the calculation (6.14), as well as multiplying the whole identity through by the factor $\langle \xi \rangle^{2s} \langle x \rangle^{2k}$ we will obtain extra terms, while arise from differentiating these factors. The new terms, in effect, have the effect of replacing the p_1 term by

$$p_1 + \langle \xi \rangle^{-2s} \langle x \rangle^{-2k} H_p\big(\langle \xi \rangle^{2s} \langle x \rangle^{2k}\big).$$

It is easy to see that this additional term is a symbol of order $(0,0)$; in fact, it is the principal symbol of $\Lambda^{-2s,-2k}[P, \Lambda^{2s,2k}]$ which has order $(0,0)$. So, effectively the difference is to replace p_1 by some other symbol $\tilde{p}_1$ of order $(0,0)$, so this allows the construction to proceed exactly as before, with b and e' redefined suitably. It would be a useful exercise to compute the exact formulae for b and e' in this setting of general orders.

6.3. **Proof of Theorem 5.3, Step 3.** At this stage we have proved the required inequality for sufficiently regular/decaying u. As with the simple example proved last lecture, this is not very satisfactory as we need to assume at least as much regularity/decay as is proved. Indeed, the $\|Bu\|^2_{H^{s,k}}$ term needs to be absorbed on the RHS, which requires it to be finite a priori. Nevertheless we have proved a quantitative estimate for sufficiently regular/decaying functions, and as with the example last lecture, we can use a regularization argument to obtain the full result.

In the simple example last time, we used $T_r = (1+r\Delta)^{-M}$ as a regularizing operator. This worked well as it commuted with the operator $P = D_{x_1}$. In the more general case, the lack of commutation is an issue and we have to be more careful. Moreover, T_r only improves in the differential sense, not the decay sense. To emphasize the point here, we will henceforth assume that γ lies in the spatial boundary, at finite frequency. Thus the appropriate 'regularization' is to multiply by $(1 + r|x|^2)^{-M}$. We will thus define a family a_r, $r \geq 0$, (again sticking to the case $s = k = 0$ for notational simplicity; note that the s parameter is actually irrelevant for us if γ is at finite frequency, and only the k is relevant) by

$$a_r = \chi_1(z_1)^2 \chi(z_1) \psi(z')^2 (1 + r|x|^2)^{-M}. \tag{6.18}$$

This will allow us to apply the regularized operators $A_r = (\mathrm{Op}_L(a_r) + \mathrm{Op}_L(a_r)^*)/2$, etc, to u (notice that although u is only assumed to be in $H^{-N,-N}$ the lack of differential regularity is not a problem as our operators are all microsupported near γ, and in particular away from frequency infinity; that is, they can be taken to be order $-\infty$ in the differential sense, so only the spatial order needs to be improved). Then, when we compute $H_p a_r + p_1 a_r$, we obtain an extra term from differentiating the regularizer. Recalling that H_p is a smooth vector field on $\overline{T^*\mathbb{R}^n}$ tangent to the boundary, in the coordinates (z_1, z', ρ_b), where $\rho_b = 1/|x|$ is a defining function for the spatial boundary, we have, in the ambient space,

$$H_p = \frac{\partial}{\partial z_1} + q(z_1, z', \rho_b)\rho_b \frac{\partial}{\partial \rho_b}$$

for some smooth function $q \in C^\infty(\overline{T^*\mathbb{R}^n})$. Therefore,

$$H_p(1 + r|x|^2)^{-M} = \frac{2Mqr|x|^2}{(1 + r|x|^2)}(1 + r|x|^2)^{-M} := f_r(1 + r|x|^2)^{-M}.$$

One can check that the factor $f_r = 2Mqr|x|^2/(1 + r|x|^2)$ is a symbol of order $(0, 0)$, uniformly as $r \to 0$. Thus, this additional term is similar to the additional term from varying the order of the symbol a: it in effect changes p_1 to a new symbol of order $(0, 0)$, although now it is r-dependent (but in a uniform way). We notice that it is proportional to M, which could be large. However, we fix M (depending on N, where $u \in H^{-N,-N}$) and *then* choose $\varGamma$ large enough so that the square root defining b_r is well-defined and smooth. It is a subtle but important technical point that we can only do a finite (although arbitrarily large) amount of regularization in this argument.

The upshot of this is that we can repeat the previous construction with r-dependent operators and obtain families of operators A_r, B_r, E'_r, R_r satisfying the identity (in the case of general orders (s, k), but assuming that γ is located away from frequency infinity)

$$i(P^* A_r - A_r P) = -\tilde{B}_r^* \tilde{B}_r - (\Lambda^{-s,-k} A_r)^* \Lambda^{-s,-k} A_r + E'_r + R_r, \tag{6.19}$$

where these operators satisfy, for every $\eta > 0$, and every $S \in \mathbb{R}$,

- $A_r \in \Psi_{\mathrm{sc}}^{S,2k}$ uniformly, and $A_r \in \Psi_{\mathrm{sc}}^{S,2k-2M}$ for each $r > 0$,
- $\tilde{B}_r \in \Psi_{\mathrm{sc}}^{S,k}$ uniformly, and $\tilde{B}_r \in \Psi_{\mathrm{sc}}^{S,k-M}$ for each $r > 0$, $\tilde{B}_r \to \tilde{B}$ in $\Psi_{\mathrm{sc}}^{S,k+\eta}$,
- $E'_r \in \Psi_{\mathrm{sc}}^{S,2k}$ uniformly, and $E'_r \in \Psi_{\mathrm{sc}}^{S,2k-2M}$ for each $r > 0$, $E'_r \to E'$ in $\Psi_{\mathrm{sc}}^{S,2k+\eta}$,
- $R_r \in \Psi_{\mathrm{sc}}^{S,2k-1}$ uniformly, and $R_r \in \Psi_{\mathrm{sc}}^{S,2k-1-2M}$ for each $r > 0$, $R_r \to R$ in $\Psi_{\mathrm{sc}}^{S,2k-1+\eta}$.

Because of the regularization, we can now take expectation values with $u \in H^{-N,-N}$ and we find that

$$\|\tilde{B}_r u\|_{L^2}^2 \le C\Big(\|GPu\|_{H^{s,k}}^2 + \langle E'_r u, u\rangle + \|u\|_{H^{-N,-N}}^2\Big), \tag{6.20}$$

We now 'take a limit' as $r \to 0$, or more precisely investigate the uniform properties of this inequality in this limit. Intuitively, we expect that the term $\langle E'_r u, u\rangle$ should be uniformly bounded, since u is in $H^{s,k}$ microlocally on $\mathrm{WF}'(E')$. This is correct, and is one of the exercises for this lecture. Thus the RHS of (6.20) is uniformly bounded, and it follows that $\tilde{B}_r u$ is uniformly bounded in L^2, as $r \to 0$. We can therefore find a sequence r_j tending to zero such that $\tilde{B}_{r_j} u$ converges weakly, say to $v \in L^2$. Similarly to last lecture, we have

$$\|v\|_{L^2}^2 \le \limsup_{r\to 0} \|\tilde{B}_r u\|_{L^2}^2 \le C\Big(\|GPu\|_{H^{s,k}}^2 + \|Eu\|_{H^{s,k}}^2 + \|u\|_{H^{-N,-N}}^2\Big). \tag{6.21}$$

On the other hand, we know that $\tilde{B}_r u$ converges to $\tilde{B}u$ in a weak Sobolev norm, since $\tilde{B}_r \to \tilde{B}$ in $\Psi_{\mathrm{sc}}^{s,k+\eta}$. So we have $\tilde{B}_r u$ converges to both $\tilde{B}u$ and v distributionally. By uniqueness of limits, $\tilde{B}u = v$ so we have the inequality (6.21) for $\tilde{B}u$ in L^2, or equivalently for Bu in $H^{s,k}$. This completes the proof of Theorem 5.3.

6.4. **Exercises.**

(1) Let $E' \in \Psi_{\mathrm{sc}}^{2s,2k}$ and suppose that $E \in \Psi_{\mathrm{sc}}^{0,0}$ is elliptic on $\mathrm{WF}'(E')$. Show that for all N there is a constant C such that, for all $u \in H^{s,k}$,
$$|\langle E'u, u\rangle| \le C\Big(\|Eu\|^2_{H^{s,k}} + \|u\|^2_{-N,-N}\Big).$$
Hint: find an operator $C \in \Psi_{\mathrm{sc}}^{0,0}$ such that $E' = (\Lambda^{s,k}E)^* C \Lambda^{s,k}E + R$, where R has sufficiently negative orders.

(2) Extend the previous result as follows: Suppose that $\sigma_L(E'_r) = \sigma_L(E')(1 + r|x|^2)^{-M}$. Prove the above inequality with E'_r in place of E' on the LHS, with a constant C independent of r for all $r \in (0, 1]$. Hint: use the fact that the operator norm of C is controlled by a suitable $S_{\mathrm{sc}}^{0,0}$-norm of its left-reduced symbol.

7. Lecture 7: Scattering calculus on manifolds (with boundary)

We introduce the scattering calculus and discuss its basic properties in this part. Some treatments here are from the lecture notes of Peter Hintz, which is available at: https://people.math.ethz.ch/∼hintzp/notes/micro.pdf .

We only deal with operators acting on scalar functions with details, but a significant portion of those general statements in this section have fairly straightforward generalization to vector bundles. Of course, difficulties in dealing with vector bundles will arise in concrete problems.

7.1. **The scattering cotangent bundle.** Let M be a n-dimensional manifold with boundary and denote its boundary by ∂M. Then its smooth structure determines a boundary defining function x. All of our bundles, symbol classes, calculi will be the same as the classical ones on manifolds without boundary on the interior of M, but not uniformly down to ∂M.

Suppose $y = (y_1, ..., y_{n-1})$ is a coordinate system near $p \in \partial M$, then
$$(x, y_1, ..., y_{n-1})$$
forms a coordinate system of M near p.

A local frame of the space[2] of vector fields that are tangent to ∂M will be
$$x\partial_x,\ \partial_{y_1},\ ...\ \partial_{y_{n-1}}. \tag{7.1}$$
We denote their $C^\infty(M)$-span by $\mathcal{V}_{\mathrm{b}}(M)$, which is called the space of b-vector fields on M. A direct computation shows that this is a Lie algebra with the Lie bracket being the standard commutator:
$$[V_1, V_2]f = V_1V_2f - V_2V_1f,\ f \in C^\infty(M). \tag{7.2}$$
Then we define the space of to be
$$\mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}(M). \tag{7.3}$$
Another characterization of $\mathcal{V}_{\mathrm{sc}}(M)$ can be given by
$$\mathcal{V}_{\mathrm{sc}}(M) = \{V \in \mathcal{V}_{\mathrm{b}}(M) : Vx = O(x^2)\}.$$

[2]Rigorously speaking, generators of the left $C^\infty(M)$-module.

From (7.1) and the definition, we know that locally $\mathcal{V}_{\mathrm{sc}}(M)$ is the span of

$$x^2\partial_x,\ x\partial_{y_1},\ \dots\ x\partial_{y_{n-1}}. \tag{7.4}$$

This is again a Lie algebra with the Lie bracket being the commutator as in (7.2). In fact, it is even better, in the sense that

$$[\mathcal{V}_{\mathrm{sc}}(M), \mathcal{V}_{\mathrm{sc}}(M)] = [x\mathcal{V}_{\mathrm{b}}(M), x\mathcal{V}_{\mathrm{b}}(M)] \subset x^2\mathcal{V}_{\mathrm{b}}(M) = x\mathcal{V}_{\mathrm{sc}}(M),$$

which means commutators has one order better decay compared with $\mathcal{V}_{\mathrm{sc}}(M)$ and this corresponds to the more general fact that the scattering calculus is commutative on the top level in both the differential and decay sense, and this is one of the major reasons that makes it more tractable[3] compared with other calculus (for example, b-calculus) when one wants to obtain Fredholm property or invertibility.

This uniquely determines[4] a vector bundle ${}^{\mathrm{sc}}TM$, which is called the *scattering tangent bundle*. The bundle of importance for our analysis is its dual bundle ${}^{\mathrm{sc}}T^*M$, which is called the *scattering cotangent bundle*. Its local frame[5] is given by

$$\frac{dx}{x^2}, \frac{dy_1}{x}, \dots \frac{dy_{n-1}}{x}. \tag{7.5}$$

This is going to be the phase space in which we work and coefficients in terms of this frame gives the 'scattering frequency'. Concretely, this means that we write the (extension of) tautological one-form as

$$\alpha = \tau\frac{dx}{x^2} + \mu\cdot\frac{dy}{x}, \tag{7.6}$$

where $\mu = (\mu_1, \dots \mu_{n-1})$ and $\mu\cdot\frac{dy}{x} = \sum_{i=1}^{n-1}\mu_i\frac{dy_i}{x}$.

In the case $M = \overline{\mathbb{R}^n}$, they are closely related to the frequency in classical Fourier analysis. Let (z, ζ) be coordinates of $T^*\mathbb{R}^n$, then one have

$$\tau\frac{dx}{x^2} + \mu\cdot\frac{dy}{x} = \zeta\cdot dz, \tag{7.7}$$

with $x = |z|^{-1}$ for $|z|$ large and $y \in \mathbb{R}^{n-1}$ is obtained from a local parametrization of $\mathbb{S}^{n-1}$. So one can think of τ as the radial component of your classical frequency while μ is the tangential component.

Of course, as we did on $\mathbb{R}^n$, to facilitate the high frequency analysis, we will compactify the fiber to be $\overline{\mathbb{R}^n}$ and denote the corresponding bundle by ${}^{\mathrm{sc}}\overline{T}^*M$. We call it the *compactified scattering cotangent bundle*, which is an n-ball bundle over M. Its boundary consists of two parts: its restriction to ∂M, which we denote by ${}^{\mathrm{sc}}\overline{T}^*_{\partial M}M$, and the 'fiber infinity', which is locally of the form $U \times \partial\overline{\mathbb{R}^n}$, where U is an open set in M on which you can trivialize ${}^{\mathrm{sc}}T^*M$. We will use ρ_{df} (df stands for 'differential face') to denote a boundary defining function of this boundary hypersurface, which is just the boundary defining function of $\overline{\mathbb{R}^n}$ formed by compactifying the $\mathbb{R}^n$ of (τ, μ). Even when M is a manifold with boundary having no corner, this will have a corner formed by the intersection of these two parts.

[3]Of course, when it is applicable!

[4]See Exercise 3.

[5]Again, by frame below, we mean the generator of a left $C^\infty(M)$-module.

Remark 7.1. If one is familiar with the so called double space construction of those calculi, then in general the conceptually correct phase space to work with should be the conormal bundle of the lifted diagonal of your double space. See [7, Lemma 4.6] for the case of b-calculus and this should be a quite general philosophical principle that applies to many settings.

7.2. **The symbol class and the quantization.** We will define the scattering symbol class and the corresponding pseudodifferential algebra in this lecture.

As we have seen in the case $M = \overline{\mathbb{R}^n}$, compared with the classical symbol class, the main property of the scattering symbol class is that it gains decay when you differentiate in spatial variables.

We will define them in terms of smooth functions on the interior of ${}^{\mathrm{sc}}\overline{T}^*M$ with certain prescribed growth or decay rate when we approach ∂M or fiber infinity, but of course, one should think of them as objects living on this compactified phase space.

Let M° be the interior of M and we denote the restriction of ${}^{\mathrm{sc}}T^*M$ to M° by ${}^{\mathrm{sc}}T^*_{M^\circ}M$. The space of scattering symbols of order (m,r), which we denote by $S^{m,r}_{\mathrm{sc}}(M)$, consists of smooth functions a on ${}^{\mathrm{sc}}T^*_{M^\circ}M$ such that in a coordinate chart:

$$|(x\partial_x)^\alpha \partial_y^\beta \partial_\tau^\gamma \partial_\mu^\delta \partial_\mu a(x,y,\tau,\mu)| \leq C_{\alpha\beta\gamma\delta} \langle \tau, \mu \rangle^{m-\gamma-|\delta|} x^{-r}. \tag{7.8}$$

We will call m the differential order, and r the decay[6] order. For fixed indices, the infimum[7] of $C_{\alpha\beta\gamma\delta}$ on the right hand side gives a seminorm on this symbol class. All those seminorms together gives the Fréchet topology on $S^{m,r}_{\mathrm{sc}}(M)$.

In the case with $M = \overline{\mathbb{R}^n}$, we know that the ∂_{z_i} will be equivalent (in the sense that the linear transformations between them have uniformly bounded coefficients on both directions) to $x^2\partial_x,\ x\partial_{y_i}$. So remain the same growth under $x\partial_x,\ \partial_{y_i}$ is indeed the same as gaining one order growth under ∂_{z_i}.

Let $\mathcal{S}(M) = \bigcap_{N\in\mathbb{N}} x^N C^\infty(M)$ be the Schwartz function class on M. Then for $a \in S^{m,r}_{\mathrm{sc}}(M)$ that is supported in a single coordinate chart, the (left) quantization of $a \in S^{m,r}_{\mathrm{sc}}(M)$ is defined to be the operator acting on functions in $\mathcal{S}(M)$ and supported in the same chart by

$$\mathrm{Op}(a)f(z) = (2\pi)^{-n} \int\int e^{i(\tau\frac{x-x'}{xx'} + \mu\cdot(\frac{y}{x} - \frac{y'}{x'}))} a(x,y,\tau,\mu) f(x',y') \frac{dx'dy'}{(x')^{n+1}} d\tau d\mu, \tag{7.9}$$

and defined on larger spaces by extension. And it act on functions with support disjoint from this chart by a kernel that is a Schwartz function on $M \times M$.

Generally, we define $\Psi^{m,r}_{\mathrm{sc}}(M)$ to be the space of operators having Schwartz kernels of the form

$$\sum_i A_i + R, \tag{7.10}$$

where A_i acts like (7.9) in various charts and $R \in \mathcal{S}(M\times M)$. That is, near the diagonal of $M \times M$, it acts by (7.9) and off the diagonal, it acts by a kernel that is a Schwartz function.

[6]In fact, this is a growth order on the symbol or operator side if one think about how the requirement changes as r increases. It will become clear that this is a good name after we define corresponding Sobolev spaces.

[7]We fix an atlas of coordinate charts that is locally finite.

This definition gives a local reduction to $\Psi_{\mathrm{sc}}^{m,r}(\overline{\mathbb{R}^n})$ modulo $\mathcal{S}(M\times M)$. Now it might be illuminating to look at the phase $\tau\frac{x-x'}{xx'}+\mu\cdot(\frac{y}{x}-\frac{y'}{x'})$ in the $\overline{\mathbb{R}^n}$ case. For example, in the region of $\overline{\mathbb{R}^n}$ such that z_n dominates all other components and $z_n>0$, then we can take

$$x=\frac{1}{z_n},\ y_i=\frac{z_i}{z_n},\ i=1,2,..n-1. \tag{7.11}$$

And we have

$$\alpha=\tau\frac{dx}{x^2}+\mu\cdot\frac{dy}{x}=(-\tau-\frac{\mu\cdot\tilde{z}}{z_n})dz_n+\sum_{i=1}^{n-1}\mu_i dz_i, \tag{7.12}$$

where $\tilde{z}=(z_1,..,z_{n-1})$. Comparing with $\alpha=\zeta\cdot dz$, we know

$$\zeta_i=\mu_i,\ 1\le i\le n-1,\ \zeta_n=(-\tau-\frac{\mu\cdot\tilde{z}}{z_n}). \tag{7.13}$$

Then we have

$$\zeta\cdot(z-z')=\mu\cdot(\frac{y}{x}-\frac{y'}{x'})+(-\tau-\mu\cdot y)(\frac{1}{x}-\frac{1}{x'}),$$

which equals to our phase to the leading order. (They differ by $(\frac{x-x'}{xx'})(\mu\cdot y)$.)

In addition, this phase is the reasonable one to use by composing elements in $\mathcal{V}_{\mathrm{sc}}(M)$ from the left. If we apply x^2D_x, xD_y, then it brings down an τ, μ factor respectively.

We will call $\Psi_{\mathrm{sc}}(M)=\bigcup_{m,l}\Psi_{\mathrm{sc}}^{m,r}(M)$ the scattering pseudodifferential algebra on M. We will discuss its structure as a multi-graded algbra in the next part. Lastly, we set

$$\Psi_{\mathrm{sc}}^{-\infty,-\infty}(M)=\bigcap_{m,l}\Psi_{\mathrm{sc}}^{m,r}(M), \tag{7.14}$$

which consists of operators with kernels being Schwartz functions on $M\times M$, and we will call them residual.

Classical symbols are defined in the same manner as the $\mathbb{R}^n$-case:

$$S_{\mathrm{sc,cl}}^{m,r}(M)=\rho_{\mathrm{df}}^{-m}x^{-r}C^\infty({}^{\mathrm{sc}}\overline{T}^*M)\subset S_{\mathrm{sc}}^{m,r}(M), \tag{7.15}$$

If a is classical, then $A=\mathrm{Op}(a)$ is called classical.

Now we turn to the quantization map. A lot of without any restriction will face the issue that you need to compare the volume growth of your manifold and the off-diagonal decay of your kernel or other objects you are summing/integrating over. In this notes we make the following assumption:

M is (topologically) compact.

The word 'topologically' means that we still allow certain 'geometrically' non-compact setting. The most typical example would be the asymptotically conic manifolds. That is, with M being a manifold with boundary as above, and equipped with a metric of the form (these are called scattering metrics)

$$g=\frac{dx^2}{x^2}+\frac{h(x,dy)}{x^2}, \tag{7.16}$$

where h is a metric on ∂M depending on x smoothly. This also gives a metric on ${}^{\mathrm{sc}}TM$, making it complete.

As we will see in Theorem 7.1, there can't be a 'perfect' quantization. So we just define it via gluing (7.9). Concretely, let $\phi_i : U_i \to \overline{\mathbb{R}^n}$ be a coordinate chart[8], and let χ_i be a partition of unity that is subordinate to this cover and let $\tilde{\chi} \in C_c^\infty(U_i)$ be identically 1 on $\mathrm{supp}\chi_i$. Then we define

$$\mathrm{Op}(a) = \sum_i \mathrm{Op}(\chi_i a)\tilde{\chi}_i, \tag{7.17}$$

where $\mathrm{Op}(\chi_i a)$ is defined via (7.9)[9] and it can act one $\tilde{\chi}_i f$ since it have support contained in U_i.

Here the localizer $\tilde{\phi}_i$ differs with ϕ_i because in (7.9) we allow f to have support slightly larger than a.

7.3. **Basic properties of the calculus.** We discuss basic properties of Ψ_{sc} in this part.

First we define the principal symbol of scattering pseudodifferential operator. Consider $A_{\mathrm{loc}} = \chi A \chi$ with $\chi \in C_c^\infty(M)$ with support contained in a single coordinate chart $\phi : U \to U' \subset \overline{\mathbb{R}^n}$ and is identically 1 on some smaller $V \subset U$ (of course, it will be 1 on $\overline{V}$ then). Then we know that A_{loc} can be identified with

$$(\phi^{-1})^* A_{\mathrm{loc}} \phi^*, \tag{7.18}$$

which is a scattering pseudodifferential operator on $\overline{\mathbb{R}^n}$ (verify this!). So locally this $(\phi^{-1})^* A_{\mathrm{loc}} \phi^*$ can be written as $\mathrm{Op}(a_{U'})$ for some $a_{U'} \in S^{m,r}(\overline{\mathbb{R}^n})$. Finally, we set

$$a_V = \tilde{\phi}^* a_{U'}|_{{}^{\mathrm{sc}}T^*_V M}, \tag{7.19}$$

where $\tilde{\phi} : {}^{\mathrm{sc}}T^*_U M \to \phi(U) \times \mathbb{R}^n$ is the trivialization induced by ϕ. And the equivalent class

$$[a_V] \in S^{m,r}_{\mathrm{sc}}(V)/S^{m-1,r-1}_{\mathrm{sc}}(V) \tag{7.20}$$

is independent of the choice of χ and the coordinate system ϕ, by checking the 'coordinate invariance' as in the classical setting[10]. This defines the principal symbol of A locally and the actual 'global' principal will be the equivalent class modulo $S^{m-1,r-1}_{\mathrm{sc}}(M)$ obtained by gluing those $a_{U'}$. It is not hard to check the following fact: if $V_1 \subset V$, then we have

$$[a_V]|_{T^*_{V_1} M} = [a_{V_1}]. \tag{7.21}$$

Definition 7.1. *The principal symbol of $A \in \Psi^{m,r}_{\mathrm{sc}}(M)$ is the unique equivalent class*

$$\sigma^{m,r}(A) \in S^{m,r}_{\mathrm{sc}}(M)/S^{m-1,r-1}_{\mathrm{sc}}(M) \tag{7.22}$$

such that: let $a \in S^{m,r}_{\mathrm{sc}}(M)$ be any representative of it and V, $[a_V]$ as above, then $[a|_{{}^{\mathrm{sc}}T^*_V M}] = [a_V]$ in $S^{m,r}_{\mathrm{sc}}(V)/S^{m-1,r-1}_{\mathrm{sc}}(V)$.

Proof. The uniqueness is easy: the restriction of $\sigma^{m,r}(A)$ to any coordinate chart is uniquely determined.

[8]Rigorously speaking, one should reduce $\overline{\mathbb{R}^n}$ to $\mathbb{R}^n$ and half-spaces further.

[9]Here we packaged pull-backs, trivialization induced by ϕ_i into the definition of Op.

[10]This is not very trivial. Read Section 6.1 of Hintz's notes if you haven't seen this.

To show the existence, we use a partition of unity $\{\phi_i\}$ subordinate to a locally finite[11] open cover $\{V_i\}$ as in the definition of $[a_V]$ above. Now we take a representative a_{V_i} for each $[a_{V_i}]$ and set

$$a = \sum_i \phi_i a_{V_i}, \quad \sigma^{m,r}(A) = [a] \in S^{m,r}_{\mathrm{sc}}(M)/S^{m-1,r-1}_{\mathrm{sc}}(M). \tag{7.23}$$

Now we verify that $[a]$ satisfies desired properties. Take any V in the definition, since for any $p \in V$, there is a $\phi \in C^\infty_c(V)$ that is identically 1 near p, so we only need to show that for all $\phi \in C^\infty_c(V)$ we have

$$[(\phi a)|_V] = [(\phi a)_V].$$

As aforementioned, making further restriction does not change the equivalent class, so we restrict $[\phi\phi_i a_{V_i}]$ and $[\phi\phi_i a_V]$ to $V_i \cap V$ to see that

$$\phi\phi_i a_{V_i} = \phi\phi_i a_V + e_i,$$

with $e_i \in S^{m-1,r-1}_{\mathrm{sc}}(M)$ and is supported in ${}^{\mathrm{sc}}T^*_{V\cap V_i}M$. Then we have

$$\phi a = \phi a_V + \sum_i e_i,$$

while $\sum_i e_i \in S^{m-1,r-1}_{\mathrm{sc}}(M)$, as desired.

□

Remark 7.2. Being able to glue those local principal symbols can be thought of a quite algebraic fact. In fact, it is easy to check $S^{m,r}_{\mathrm{sc}}(M)$ and $S^{m-1,r-1}_{\mathrm{sc}}(M)$ are sheaves on M. But in general quotient of sheaves is only a pre-sheaf. However, $S^{m-1,r-1}_{\mathrm{sc}}(M)$ is a flasque (that is, the restriction map is surjective) subsheaf of $S^{m,r}_{\mathrm{sc}}(M)$, which is enough to derive that $S^{m,r}_{\mathrm{sc}}(M)/S^{m-1,r-1}_{\mathrm{sc}}(M)$ is a sheaf. And we can verify compatibility conditions for $[a_V]$ in terms of intersection and restriction to derive that this global principal symbol exists. In the proof above, $S^{m-1,r-1}_{\mathrm{sc}}(M)$ being flasque is reflected for example by saying e_i is global directly.

Now we turn to the ellipticity. For $a \in S^{m,r}_{\mathrm{sc}}(M)$, it is called elliptic (in $S^{m,r}_{\mathrm{sc}}(M)$) at $q \in \partial({}^{\mathrm{sc}}\overline{T}^*M)$ if

$$|x^r \rho^m_{\mathrm{df}} a(x,y,\tau,\mu)| \geq C > 0$$

in a neighborhood of q. We say a is elliptic (in $S^{m,r}_{\mathrm{sc}}(M)$) if it is elliptic at every point of $\partial({}^{\mathrm{sc}}\overline{T}^*M)$. Of course, this property descends to a property of $S^{m,r}_{\mathrm{sc}}(M)/S^{m-1,r-1}_{\mathrm{sc}}(M)$ and we say A is elliptic if $\sigma^{m,r}_{\mathrm{sc}}(A)$ is elliptic. This is equivalent to the following fact: $a \in S^{m,r}_{\mathrm{sc}}(M)$ is elliptic if and only if there exists $b \in S^{-m,-r}_{\mathrm{sc}}(M)$ such that

$$ab - 1 \in S^{-1,-1}_{\mathrm{sc}}(M). \tag{7.24}$$

Also, A is elliptic in $\Psi^{m,r}_{\mathrm{sc}}(M)$ if and only if there is a $B \in \Psi^{-m,-r}_{\mathrm{sc}}(M)$ such that

$$AB - \mathrm{Id},\ BA - \mathrm{Id} \in \Psi^{-1,-1}_{\mathrm{sc}}(M). \tag{7.25}$$

In fact, the error term can be improved to be in $\Psi^{-\infty,-\infty}_{\mathrm{sc}}(M)$ and such B is called the parametrix of A.

[11]It can be taken to be finite by our assumption.

Proposition 7.1. *The principal symbol map gives the following short exact sequence*[12]*:*

$$0 \to \Psi_{\mathrm{sc}}^{m-1,r-1}(M) \to \Psi_{\mathrm{sc}}^{m,r}(M) \xrightarrow{\sigma^{m,r}(\cdot)} S_{\mathrm{sc}}^{m,r}(M)/S_{\mathrm{sc}}^{m-1,r-1}(M) \to 0.$$

This means that $\sigma^{m,r}(\cdot)$ captures the leading order behaviour in both the differential sense and the decay sense. If two operators have the same principal symbol, then the coincide up to $\Psi_{\mathrm{sc}}^{m-1,r-1}(M)$-level (in the sense that their difference lies in here).

Then the proof of the composition law can be done by local reduction to the $\overline{\mathbb{R}^n}$-case. We state the conclusion below.

Proposition 7.2. Ψ_{sc} *is a multi-graded $*$-algebra in the following sense (let $*$ denote the formal adjoint)*

$$\begin{aligned} A \in \Psi_{\mathrm{sc}}^{m,r}(M) &\implies A^* \in \Psi_{\mathrm{sc}}^{m,r}(M), \\ A \in \Psi_{\mathrm{sc}}^{m_1,r_1}(M),\ B \in \Psi_{\mathrm{sc}}^{m_2,r_2}(M) &\implies A \circ B \in \Psi_{\mathrm{sc}}^{m_1+m_2,r_1+r_2}(M). \end{aligned} \tag{7.26}$$

In addition, the principal symbol map 'preserves products':

$$\sigma^{m_1+m_2,r_1+r_2}(AB) = \sigma^{m_1,r_1}(A)\sigma^{m_2,r_2}(B). \tag{7.27}$$

Of course, the product on the right hand side is defined via taking representatives to form a product and then take the equivalence class again. One can check that this does not depend on the choice of representatives.

As we will see in the proof of propagation estimates, commutators will play a very important role in microlocal analysis. In scattering calculus, commutators has the following property that is analogous to the classical case:

Proposition 7.3. *Let $A \in \Psi_{\mathrm{sc}}^{m_1,r_1}(M),\ B \in \Psi_{\mathrm{sc}}^{m_2,r_2}(M)$, then $[A,B] \in \Psi_{\mathrm{sc}}^{m_1+m_2-1,r_1+r_2-1}(M)$ and*

$$\sigma^{m_1+m_2-1,r_1+r_2-1}([A,B]) = -i\{\sigma^{m_1,r_1}(A), \sigma^{m_2,r_2}(B)\}. \tag{7.28}$$

The proof comes down to local reduction to the $\mathbb{R}^n$ case and use the asymptotic expansion. The Poisson bracket on the right hand side is defined via taking representatives and compute using the definition that is almost the same as the $\mathbb{R}^n$-case which we recall below. For $p, a \in C^\infty({}^{\mathrm{sc}}T^*M)$, $\{p,a\}$ is defined via

$$\{p,a\} = H_p a, \tag{7.29}$$

where H_p is the Hamilton vector field uniquely determined by

$$dp(H') = \omega_{\mathrm{sc}}(H_p, H'), \tag{7.30}$$

for any smooth vector field H' on ${}^{\mathrm{sc}}T^*M$. Here ω_{sc} is the symplectic form

$$\begin{aligned} \omega_{\mathrm{sc}} &= -d(\tau\frac{dx}{x^2} + \mu \cdot \frac{dy}{x}) \\ &= -d\tau \wedge \frac{dx}{x^2} - d\mu \wedge \frac{dy}{x} + x\frac{dx}{x^2} \wedge \frac{\mu \cdot dy}{x}. \end{aligned} \tag{7.31}$$

Of course, both of the Poisson bracket and the Hamiltonian vector field are the same as the classical one in the interior and one can compute them in terms of local coordinates. But notice that our contact form is *NOT* $\tau dx + \mu \cdot dy$, but there are

[12] Say, as left $C^\infty(M)$-modules. But this is not that important in our analysis.

x-factors involved and the expression in terms of those coordinates will look different from the classical case.

By condition (7.30) solving for the undetermined coefficients, we have

$$\begin{aligned} H_p =&\partial_\tau p(x^2\partial_x) + \partial_\mu p(x\partial_y) - (x^2\partial_x p + x\mu\cdot\partial_\mu p)\partial_\tau \\ &- (x\partial_y p - x\mu)\cdot\partial_\mu. \end{aligned} \tag{7.32}$$

One point we would like to emphasize here is that one can see from (7.28) that Hamiltonian dynamics will enter naturally if one want to

In fact, one can check that this Poisson bracket satisfies the Jacobi's identity and makes $C^\infty({}^{\mathrm{sc}}T^*M)$ a Lie algebra. On the other hand $[\cdot,\cdot]$ also makes Ψ_{sc} a Lie algebra. Then (7.28) says that the principal symbol map (modulo the i-factor) preserves those structures on the top level. Of course, one would like to ask can we preserve this completely precisely? The answer is no, even in a much more limited setting. This is the Groenewold's theorem:

Theorem 7.1. *On $\mathbb{R}^n$, as long as $S \subset C^\infty(T^*\mathbb{R}^n)$ includes symbols that is polynomial in positions and frequencies, then there is no quantization map $Q : S \to \Psi(\mathbb{R}^n)$ such that*

$$-iQ(\{f,g\}) = [Q(f), Q(g)]. \tag{7.33}$$

A sketch of the proof is as following: the condition for polynomials of first two orders determines it have to be the Weyl quantization. And then the quantized version of (x_1, ξ_1 stands for the first component of position and frequency respectively)

$$x_1^2\xi_1^2 = \frac{1}{9}\{x_1^3, \xi_1^3\} = \frac{1}{3}\{x_1^2\xi_1, x_1\xi_1^2\}$$

will lead to contradiction. See [2, Section 13.4] for details.

7.4. **Sobolev spaces, mapping properties.** Recall we have the Schwartz function class on M defined by

$$\mathcal{S}(M) = \bigcap_{N\in\mathbb{N}} x^N C^\infty(M), \tag{7.34}$$

and it is equipped with the Fréchet topology induced by this intersection (for example: $\sup_{(x,y)\in\phi(U)} |x^{-N}\partial^\alpha_{x,y}u|$ for a fixed set of charts (U,ϕ) forming an finite open cover of M).

Then we denote $\mathcal{S}'(M)$ to be the dual space of it (continuous linear functionals on it). It is called the space of tempered distributions on M.

The metric (7.16) gives a volume form $|\frac{dxdy}{x^{n+1}}|$ and this defines $L^2_{\mathrm{sc}}(M)$. Concretely, let χ_i be a partition subordinate to coordinate charts $\{(U_i,\phi_i)\}$ consists of finitely U_i and we define

$$\|u\|_{L^2_{\mathrm{sc}}(M)} = (\sum_i \int |\chi_i u(x,y)|^2 |d\nu|)^{1/2} \tag{7.35}$$

for $u \in \mathcal{S}(M)$. Here $|d\nu|$ is $|\frac{dxdy}{x^{n+1}}|$ if we are near the boundary and if some of U_i is completely away from ∂M and have coordinate system $z = (z_1, .., z_n)$, then we just use $|dz|$. One can verify that this is a norm and $L^2_{\mathrm{sc}}(M)$ is defined to be the completion of $\mathcal{S}(M)$ with respect to this norm.

Let $\{(U_i, \phi_i, \chi_i)\}$ be as above, and set

$$\|u\|_{H_{\mathrm{sc}}^{m,r}(M)} = \big(\sum_i \|(\phi_i^{-1})^*(\chi_i u)\|^2_{H_{\mathrm{sc}}^{m,r}(\overline{\mathbb{R}^n})}\big)^{1/2}. \tag{7.36}$$

for $u \in \mathcal{S}(M)$, and again set $H_{\mathrm{sc}}^{m,r}$ to be the completion of $\mathcal{S}(M)$ with respect to this norm. This is a Hilbert space with the inner product defined by

$$\langle u, v\rangle = \sum_i \langle A_i(\phi_i^{-1})^*(\chi_i u), A_i(\phi_i^{-1})^*(\chi_i v)\rangle, \tag{7.37}$$

where A_i is a invertible elliptic operator in $\Psi_{\mathrm{sc}}(\overline{\mathbb{R}^n})$ and $\langle\cdot,\cdot\rangle$ is the L^2-pairing with respect to $|dz|$ or $|\frac{dxdy}{x^{n+1}}|$ as above, depending on we are near the boundary or not.

The boundedness of Ψ_{sc}-operators on Sobolev spaces are proved by reducing to the $\overline{\mathbb{R}^n}$ case and the residual part with $\mathcal{S}(M\times M)$ kernel is easy to deal with. We only list results below for the convenience of reference afterwards.

Proposition 7.4. *Let $A \in \Psi_{\mathrm{sc}}^{m,r}(M)$, then for any $s, l \in \mathbb{R}$, A is a continuous map from $H_{\mathrm{sc}}^{s,l}(M)$ to $H_{\mathrm{sc}}^{s-m,l-r}(M)$:*

$$\|Au\|_{H_{\mathrm{sc}}^{s-m,l-r}(M)} \le C\|u\|_{H_{\mathrm{sc}}^{s,l}}, \tag{7.38}$$

where the constant C may depend on A, m, r, s, l and choices we made in the definition of norms.

As aforementioned, $A \in \Psi_{\mathrm{sc}}^{m,r}(M)$ is elliptic if and only if there is a $B \in \Psi_{\mathrm{sc}}^{-m,-r}(M)$ such that

$$AB - \mathrm{Id},\ BA - \mathrm{Id} \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}(M). \tag{7.39}$$

And this yields the elliptic estimate:

Proposition 7.5. *Suppose $A \in \Psi_{\mathrm{sc}}^{m,r}(M)$ is elliptic, then (7.38) can almost be reversed with an error term:*

$$\|u\|_{H_{\mathrm{sc}}^{s,l}} \le C\big(\|Au\|_{H_{\mathrm{sc}}^{s-m,l-r}(M)} + \|u\|_{H_{\mathrm{sc}}^{-N,-N}}\big). \tag{7.40}$$

Here $N \in \mathbb{R}$ is arbitrary, but the useful case is when N is large: $-N < s, l$.

Finally, we briefly introduce the scattering the wavefront sets for distributions and operators. Let $A \in \Psi_{\mathrm{sc}}^{m,r}(M)$, then $\mathrm{WF}'_{\mathrm{sc}}(A)$ is the subset in $\partial({}^{\mathrm{sc}}\overline{T}^*M)$ that captures locations and frequencies where A is non-trivial, where trivial means acting with integral kernel that is a Schwartz function. Concretely, let a be the full symbol of A (potentially modulo a $\Psi_{\mathrm{sc}}^{-\infty,-\infty}$-term, according to our definition of the operator class) using the quantization (7.17), then for $q \in \partial({}^{\mathrm{sc}}\overline{T}^*M)$, we say that $q \notin \mathrm{WF}'_{\mathrm{sc}}(A)$ if there exists $\chi \in C^\infty({}^{\mathrm{sc}}\overline{T}^*M)$ such that $\chi(q) = 1$ and $\chi a \in S_{\mathrm{sc}}^{-\infty,-\infty}(M)$, which is the space of Schwartz functions on ${}^{\mathrm{sc}}\overline{T}^*M$. This is equivalent to that there is a $Q \in \Psi_{\mathrm{sc}}^{0,0}$ that is elliptic at q such that $QA \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$, which means A is acting with an integral kernel that is a Schwartz function microlocally at q

For $u \in \mathcal{S}'(M)$, the scattering wavefront set of order s, l, denoted by $\mathrm{WF}_{\mathrm{sc}}^{s,l}(u)$ is the subset of $\partial({}^{\mathrm{sc}}\overline{T}^*M)$ that captures locations and frequencies where u fails (to have enough regularity of decay) to lie in $H^{s,l}_{\mathrm{sc}}(M)$. Let $q \in \partial({}^{\mathrm{sc}}\overline{T}^*M)$, we say $q \notin \mathrm{WF}^{s,l}_{\mathrm{sc}}(u)$ if there is a $Q \in \Psi^{0,0}_{\mathrm{sc}}$ that is elliptic at q such that $Qu \in H^{s,l}_{\mathrm{sc}}(M)$. One can also take union over those finite order wavefront sets to form a set that measure where u fails trivial (i.e., being Schwartz):

$$\mathrm{WF}_{\mathrm{sc}}(u) = \bigcup_{s,l\in\mathbb{N}} \mathrm{WF}^{s,l}_{\mathrm{sc}}(u).$$

7.5. **Generalization to vector bundles.** The scattering algebra and pseudodifferential algebras constructed in previous chapters extends to operators between sections of vector bundles naturally. We describe this process very briefly for the scattering algebra as given in [8, Section 3], and this transplant to other algebras in a verbatim manner. The major motivation of this generalization is the application to systems of PDEs.

Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles over M of rank r_E and r_F respectively, either complex or real. We define the space of scattering pseudodifferential operators from E to F with differential order m and decay order l as

$$\Psi^{m,l}_{\mathrm{sc}}(M; E, F) := C^\infty(X; \mathrm{Hom}(E, F)) \otimes \Psi^{m,l}_{\mathrm{sc}}(M), \tag{7.41}$$

where the tensor product is over $C^\infty(M)$, viewing $C^\infty(X; \mathrm{Hom}(E, F))$ as a right $C^\infty(M)-$module and $\Psi^{m,l}_{\mathrm{sc}}(M)$ as a left $C^\infty(M)-$module. Equivalently, an element in $\Psi^{m,l}_{\mathrm{sc}}(M; E, F)$ is a matrix with entries in $\Psi^{m,l}_{\mathrm{sc}}(M)$. Suppose $e_1, ..., e_{r_E}$ and $f_1, ..., f_{r_F}$ are local frames of E and F over an open set O respectively, then for any $K \Subset U$, we have $P_{ij} \in \Psi^{m,l}_{\mathrm{sc}}(M), 1 \le j \le r_E, 1 \le i \le r_F$ such that

$$P(\sum_{j=1}^{r_E} \phi_j e_j) = \sum_{i,j} P_{ij}(\phi_j) f_i, \tag{7.42}$$

where $\phi_i \in C^\infty_c(O)$ and $\mathrm{supp}\phi_i \subset K$. We transposed the matrix interpretation compared with notations in [8] to make it more compatible with basic linear algebra.

The space of corresponding symbols, denoted by $S^{m,l}_{\mathrm{sc}}(M; E, F)$ is the space of $r_F \times r_E$ matrices with entries in $S^{m,l}_{\mathrm{sc}}(M)$, and the quantization map sending a symbol to $\Psi^{m,l}_{\mathrm{sc}}(M; E, F)$ is applying the quantization map in the scalar case componentwise. The ellipticity

The Sobolev spaces $H^{s,r}_{\mathrm{sc}}(M; E)$ is defined using a partition of unity, a local trivialization of E, and a scattering connection ∇^{sc}, which gives the notion of differentiating sections of E along scattering vector fields, i.e., sending sections of E to sections of E satisfying conditions for connections on vector bundles, but with $\mathcal{V}(M)$ replaced by $\mathcal{V}_{\mathrm{sc}}(M)$. For a positive integer s, $H^s_{\mathrm{sc}}(M; E)$ is defined to be the space of sections of E such that it up to $s-$order derivatives using ∇^{sc} and scattering vector fields are square integrable. Here we assume there is a fixed Hermitian inner product (and hence volume form) on E to define the integration. For general $s \in R$, we define for $s \ge 0$ by interpolation, and then use duality to define the $s < 0$ case. The weighted case is done by multiplying x^r componentwise, where x is the boundary defining function of M.

The concept of the adjoint operator requires us to fix a density Ω, and define A^* of $\Psi_{\mathrm{sc}}^{m,l}(M;E,F)$ as an element in $\Psi_{\mathrm{sc}}^{m,l}(M;F^*\otimes\Omega,E^*\otimes\Omega)$ using the equation of pairing

$$\langle Au,v\rangle=\overline{\langle u,A^*v\rangle}, \tag{7.43}$$

where u,v are sections of E and $F^*\otimes\Omega$ respectively. $a\in S_{\mathrm{sc}}^{m,l}(M;E,F)$ is said to be elliptic at (z,ζ_{sc}) if on a conic neighborhood of this point there exists $b\in S_{\mathrm{sc}}^{-m,-l}(M;F,E)$ such that $b\circ a-\mathrm{Id}_E\in S_{\mathrm{sc}}^{-1,-1}(M;E,E)$ and $a\circ b-\mathrm{Id}_F\in S_{\mathrm{sc}}^{-1,-1}(M;F,F)$. Then the principal symbol the mapping properties, elliptic estimates, wavefront sets are similar to the scalar case.

Exercises.

(1) Check the claim we made in Section 3: when $M=\overline{\mathbb{R}^n}$, ∂_{z_i} will be equivalent (in the sense that the linear transformations between them have uniformly bounded coefficients on both directions) to $x^2\partial_x,\ x\partial_{y_i}$.
(2) Check the claim that a scattering metric as in (7.16) gives a metric on ${}^{\mathrm{sc}}TM$ and it is complete. Can we do something similar to ${}^{\mathrm{sc}}T^*M$?
(3) In Section 3, we claimed that taking $\mathcal{V}_{\mathrm{sc}}(M)$ as local sections uniquely determines a vector bundle. Verify this. (Hint: For any $p\in M$, let $\mathcal{I}_p\subset C^\infty(M)$ be the ideal of smooth functions vanishing at p, then define the fiber

$${}^{\mathrm{sc}}T_pM=\mathcal{V}_{\mathrm{sc}}(M)/\mathcal{I}_p\cdot\mathcal{V}_{\mathrm{sc}}(M).$$

Then verify that this gives a vector bundle. Maybe think about what is $C^\infty(M)/\mathcal{I}_p$ first.)
(4) Verify the equivalence of two definitions of the ellipticity given in Section 7.3.
(5) Derive (7.32).
(6) In terms of the quantization map in (7.17), verify: $\mathrm{Op}(1)=\mathrm{Id}$. And this quantization map is almost surjective: $\Psi_{\mathrm{sc}}^{m,r}(M)=\mathrm{Op}(S_{\mathrm{sc}}^{m,r}(M))+\Psi_{\mathrm{sc}}^{-\infty,-\infty}(M)$.

8. Lecture 8: Applications to Inverse problems

In this part we will discuss some application of the scattering calculus to inverse problems. In particular, we will discuss the result of Uhlmann and Vasy (with an Appendix by Zhou) [11], but with a slightly different proof, which is more similar to the presentation in later papers [10] [15] [5] [14]. But we avoid introducing a quasi-homogeneous semiclassical calculus, which is used in some of those referred papers.

We can't offer a thorough literature review, but only refer to [10, Section 1] [12].

8.1. The X-ray transform.

Let (X,g) be a Riemannian manifold with boundary. The geodesic X-ray transform is a generalization of the Radon transform, and the inverse problem on it can be formulated as follows: On a Riemannian manifold (X,g), the information we have are integrals like

$$(I_0f)(\gamma(\cdot)):=\int_\gamma f(\gamma(t))dt, \tag{8.1}$$

where γ is a geodesic segment in a neighborhood O_p of a fixed point $p\in\partial X$. Here 0 stands for viewing f as a tensor of rank 0. And the general case is defined via taking the integrand to be $f(\gamma(t))(\dot\gamma(t),...,\dot\gamma(t))$ when f is a tensor field.

In this part, we consider the local geodesic ray transform with weights in dimension at least 3. 'Local' means that the geodesic segment we integrate over lies in O_p and has endpoints on ∂X, see Section 8.2.3 for the more detailed definition. We show that we can recover the restriction of the function f to O_p, which amounts to the injectivity of I_0, by proving an estimate which uses the Sobolev norm of $I_0 f$ (viewed as a function on the projective sphere bundle PSX) to control the Sobolev norm of f. In addition, this estimate is stable under small perturbations of the metric g.

8.2. **Notations and results.**

8.2.1. *The set up.* Let (X, g) be a Riemannian manifold with boundary. It is convenient to consider a larger region containing X. So suppose X is embedded as a strictly convex domain in a Riemannian manifold $(\tilde{X}, g)$ (we have used the same notation to indicate the smooth extension of the metric). Here convexity means when a geodesic is tangent to ∂X, it is tangent and curving away from X.

Concretely, let $\bar{X}$ be the closure of X in $\tilde{X}$ and let ρ be the boundary defining function of $\tilde{X}$, which means $\rho(z)$ vanishes on ∂X, $\rho(z) > 0$ on X, and satisfy the non-degeneracy condition $d\rho \neq 0$ when $\rho = 0$. Using G to denote the dual metric function on $T^*\tilde{X}$, the convexity means that if at some $\beta \in T^*_p\tilde{X}\backslash o$ with $p \in \partial X$ and o being the zero section, we have:

$$\text{if } (H_G\rho)(\beta) = 0, \text{ then } (H_G^2\rho)(\beta) < 0. \tag{8.2}$$

We will consider local geodesic transform near p in a neighborhood $O_p \subset U$ of p in X. See Section 8.2.3 for more details on O_p and the meaning of local here. Recall that an initial point and an tangent vector at this point determine a geodesic. The bundle we use to parametrize geodesics is the *projective sphere bundle*, denoted by PSX, whose fibers are $\mathbb{R}\times\mathbb{S}^{n-2}$. It parametrizes geodesics whose initial velocities has unit tangential component, except for those ones that are normal to our foliation (corresponding to $\lambda = \pm\infty$). Those excluded geodesics are irrelevant for our purpose since our cut-off χ is restricting our analysis to those geodesics that are almost tangential to our foliation.

8.2.2. *The foliation condition and the choice of the coordinate system.* We first introduce another boundary defining function $\tilde{x}$ satisfying

$$d\tilde{x}(p) = -d\rho(p),\ \tilde{x}(p) = 0, \tag{8.3}$$

whose level sets are strictly convex from the sub-level sets $\{\tilde{x} < -T\}$ for a constant $T > 0$, which means geodesics tangential to this region will curve away from it. This function is used to introduce the artificial boundary, which is a level set of $\tilde{x}$: $\{\tilde{x} = -c\}$ for $c > 0$. This level set intersects with ∂X and together with ∂X it encloses a small region on which our discussion happens. This allows us to conduct analysis locally. In terms of this new parameter $c > 0$, the region O_p is

$$\Omega_c := \{z \in X : \tilde{x}(z) \geq -c, \rho(z) \geq 0\}. \tag{8.4}$$

We can choose $\tilde{x}$ such that $\bar{\Omega}_c$ is compact for c sufficiently small. Our proof for the local result is valid for all small c.

We give an explicit construction of $\tilde{x}$ here to show it exists locally (thus can be used for our Theorem 8.1), but our result is valid for any $\tilde{x}$ satisfying conditions above.

Shrinking O_p if necessary, we can assume the neighborhood we are working on is entirely in a local coordinate patch. We take

$$\tilde{x}(z) = -\rho(z) - \epsilon|z-p|^2, \quad z \in O_p, \tag{8.5}$$

where $|\cdot|$ means the Euclidean norm in this coordinate patch, and this term is introduced to enforce the region characterized by $\tilde{x}$ to be compact. Here $\epsilon > 0$ is a fixed constant chosen before we choose c. For example, we can take $\epsilon = 1$. Taking $c > 0$ sufficiently small, $\{\tilde{x} > -c\}$ is compact. This is because $\tilde{x} > -c, \rho \geq 0$ implies $\rho \leq C$ and $|z-p| \leq c/\epsilon$. Since we are in a fixed coordinate patch, topologically this region is a closed subset of a compact Euclidean ball, hence it is compact. In addition, by the discussion in [11, Section 3.1], each Σ_t with $0 \leq t \leq c$ is convex in the sense that any geodesic tangent to it curves away from $\{\tilde{x} \leq -t\}$.

If we define Ω_c to be $\{0 \leq \rho(z) \leq c\}$, the region might be non-compact (even when c is small, it might be a long thin strip near the boundary). So we use a modification of $-\rho$ making the level sets less convex to enforce its intersection with ∂X happen in a compact region.

We now turn to the *convex foliation condition* we need. From now on, we assume $\tilde{x}$ to be any function that satisfies (8.3) and convexity condition after it. Our foliation of the part of X near ∂X is given by level sets of $\tilde{x}$. That is, the family of hypersurfaces $\{\tilde{\Sigma}_t = \tilde{x}^{-1}(-t), 0 \leq t \leq T\}$. Here we choose T to be a number such that desired properties of $\tilde{x}$ hold from $\tilde{\Sigma}_0$ up to $\tilde{\Sigma}_T$. For the Theorem 8.1, which concerns the local injectivity, we only use the part of the foliation with $t \leq c$. While for the Corollary, which concerns the global result, we use the entire foliation up to $\tilde{x} = -T$. By our choice of c, we may take $T = c$. Taking T larger will make the region on which our result holds larger. In fact, one may apply a layer stripping method to obtain injectivity result up to $\tilde{\Sigma}_T$. When $T > c$, we may take $\tilde{\Sigma}_c$ as the 'new boundary' and apply our Theorem 8.1, and then repeat. For more details of the layer stripping method, see the discussion after [11, Corollary].

Finally, the coordinate system we use is

$$(x, y, \xi, \eta), \tag{8.6}$$

where $x = \tilde{x} + c$, and y is the coordinate on $\tilde{\Sigma}_t$, which are line segments in our context. ξ, η are fiber variables dual to x, y respectively in the scattering cotangent bundle ${}^{sc}T^*X$ (i.e., we are writing covectors as $\xi\frac{dx}{x^2} + \eta\frac{dy}{x}$).

We emphasize that introducing $\tilde{X}$ and the artificial boundary $\{\tilde{x} = -c\}$ brings us convenience in this framework, allowing us to restrict our analysis in to this local region and use the scattering pseudodifferential calculus.

8.2.3. *The local geodesic ray transform.* Geodesics below are with respect to the metric g. Recalling (8.4), we replace $\tilde{x}$ by $x = \tilde{x} + c$, so that x itself becomes the defining function of the artificial boundary. In an open set $O \subset \bar{X}$, for a geodesic segment $\gamma \subset O$, we call it *O-local geodesic* if its endpoints are on $\partial X \cap O$, and **all geodesic segments we consider below are assumed to be O_p-local** with O_p as in the previous part.

As we mentioned in the introduction, *the local geodesic ray transform* of a function f is defined by:

$$I_0 f(s) := \int_\gamma f(\gamma(t))dt,$$

where $s \in PSX$, $\gamma(\cdot)$ is the geodesic determined by s. So our geodesic ray transform is a function on PSX.

8.2.4. *The main result.* We use exponentially weighted Sobolev spaces: $H^s_F(O_p) := e^{\Phi_F(x)}H^s(O_p) = \{f \in H^s_{loc}(O_p) : e^{-\Phi_F(x)}f \in H^s(O_p)\}$, where the additional subscript F is a positive constant, which indicates the exponential conjugation, $\Phi_F(x) = \frac{F}{x}$ when x is close to 0, and $\Phi_F(x) = \frac{F}{c-x}$ when x is close to c.

For exponentially weighted Sobolev spaces on other manifolds, we use the same notation with O_p replaced by that manifold. Furthermore, $PSX|_{O_p}$ is the restriction of the projective sphere bundle to O_p. With all these preparations, the main theorem is:

Theorem 8.1. *For $p \in \partial X$, with $\tilde{x}$ as above, we can choose $O_p = \{\tilde{x} > -c\} \cap \bar{X}$, so that the local geodesic transform is injective on $H^s(O_p)$, $s \geq 0$. More precisely, there exists $C > 0$ such that for all $f \in H^s_F(O_p)$,*

$$||f||_{H^s_F(O_p)} \leq C||I_0 f||_{H^{s+1}(PSX|_{O_p})}. \tag{8.7}$$

In the corollary below, $X, \tilde{\Sigma}_t$ are defined as above, and in addition we assume that $\bar{X}$ is compact.

Corollary 8.1. *If the convex foliation construction $\{\tilde{\Sigma}_t\}$ is valid up to $\{\tilde{x} = -T\}$ and $K_T := X \setminus \cup_{t\in[0,T)} \tilde{\Sigma}_t$ has measure zero, the global geodesic X-ray transform is injective on $L^2(X)$. If K_T has empty interior, the global geodesic transform is injective on $H^s_F(O_p)(X)$ for $s > \frac{n}{2}$.*

Remark 8.1. We added 'global' because the function are not restricted to O_p anymore. In addition, our result is stable since all the conditions we need in the proof are also satisfied by small perturbations of g, and the constant C in (8.7) can be made uniform for small perturbations of g, hence the same result holds for small metric perturbations.

Proof. Assuming the theorem holds, we prove the corollary. For nonzero $f \in L^2(X)$ and K_T has measure zero case, supp$f$ has non-zero measure by the definition of $L^2(X)$. Consider $\tau := \inf_{\mathrm{supp} f}(-\tilde{x})$. If $\tau \geq T$, then supp$f \subset K_T$, which has measure zero, contradiction. So $\tau < T$ and by definition $f \equiv 0$ on $\tilde{\Sigma}_t$ with $t < \tau$. By the definition of τ, closedness of supp$f$ and compactness of $\bar{X}$, we know there exists $q \in \tilde{\Sigma}_\tau \cap \mathrm{supp} f$. However, consider the manifold given by $\{\tilde{x} < -\tau\}$, to which we can apply our theorem. Since we have local injectivity near q, we conclude that q has a neighborhood disjoint with supp$f$, contradiction.

If $f \in H^s(X), s > \frac{n}{2}$, $f \neq 0$, then f is continuous by the Sobolev embedding theorem and consequently supp$f$ has non-empty interior since .Then apply local result to a fixed point in supp$f$ gives the contradiction. □

8.3. **Why scattering calculus?** We briefly explain why we need to use the scattering calculus instead of Kohn-Nirenberg or Hörmander type calculus here. Before the result of Uhlmann and Vasy [11], Stefanov and Uhlmann [9] have already shown that the normal operator we define below (without the cutoff) is an elliptic pseudodifferential operator.

However, if one don't restrict to geodesics that are becoming more and more tangent to the boundary of the local region, then the right hand side of the stability estimate will involve Sobolev norms on a slightly larger region, since the X-ray transform and in turn the normal operator will involve information outside this region. Consequently, this does not give the desired invertibility.

Now if we introduce the cut-off, as we do below like $\chi(\frac{\lambda}{x})$ with $\chi(\cdot) \in C_c^\infty(\mathbb{R})$, then they are not bounded under actions of D_x and the resulting normal operator is not a classical PsiDO as one approaches the boundary (not the ∂X, but the one introduced by localization). But they are bounded under repeated applications of xD_x and the normal operator lies in Ψ_{sc}.

In addition, this localization is necessary as one will see in the proof. The smallness of the region works like a 'semiclassical' parameter the finally allows us to remove the error term in the elliptic estimate.

Of course, one can choose other scaling to restrict to geodesics tangent to the artificial boundary. For example one can use $\sqrt{x}$ here as the defining function of the artificial boundary and consider geodesics with $|\lambda| \lesssim \sqrt{x}$. But that results in some pseudo-differential algebras in which one needs to treat non-commutative (operator-valued) boundary symbols. See [11, Remark 3.4].

8.4. **The pseudodifferential property and ellipticity of the normal operator.** In this section we prove the pseudodifferential property and ellipticity of the exponentially conjugated microlocalized normal operator. The exponential conjugation is needed because although the Schwartz kernel of A in the previous section behaves well when $X = \frac{x'-x}{x^2}$, $Y = \frac{y'-y}{x}$ are bounded, it is not so when $(X, Y) \to \infty$. This conjugation gives additional exponential decay to resolve this issue.

Using the notation $(z, \nu) = (x, y, \lambda, \omega) \in PSX$, we define the modified adjoint operator of I_0 as

$$(Lv)(z) := x^{-2} \int \chi(\frac{\lambda}{x}, y) v(\gamma_{x,y,\lambda,\omega}) d\lambda d\omega,$$

where $\gamma_{x,y,\lambda,\omega}(t)$ is the geodesic starting at (x, y) with initial tangent vector $(\lambda, \omega), \omega = \pm 1$. χ is smooth and compactly supported in the first variable, with $\chi(0, y) = 1$ and the size of its support is uniformly bounded. All derivatives with respect to y are also uniformly bounded. v is a function defined on the space of O_p-local geodesic segments, whose prototype is the geodesic ray transform

$$v(\gamma) = I_0 f(\gamma) = \int_\gamma f(\gamma(t)) dt,$$

in which $f(\gamma(t))$ can be replaced by higher order tensors, coupling with $\dot{\gamma}(t)$ in all of its slots in more general situations. By the compactness of $\bar{\Omega}_c$ discussed after (8.5) and $|(\lambda, \pm 1)| \geq 1$ and the convexity assumption on we made, [14, Lemma 3.1] shows that there exits a uniform bound T_g of the escape time of $\bar{O}_p$. Thus we assume $|t| \leq T_g$ in arguments below. I_0 is the original geodesic ray transform operator and L is its adjoint if we ignore χ and assume fast decay conditions on integrands. So their composition is the model of the normal operator.

We define the conjugated normal operator A_F as

$$A_F = e^{-F/x} L \circ I_0 e^{F/x}.$$

By the definition of L and I_0, we know A_F acts by

$$A_F f(z) := x^{-2} \int e^{-\frac{F}{x} + \frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} \chi(\frac{\lambda}{x}, y) f(\gamma_{x,y,\lambda,\omega}(t)) dt |d\nu|, \tag{8.8}$$

with $|d\nu| = |d\lambda d\omega|$ being a smooth density.

Recall the parameter c in the definition $O_p = \{\tilde{x} > -c\} \cap \bar{X}$ and we only concern the region $0 \leq x = \tilde{x} + c \leq c$, we have:

Theorem 8.2. *$A_F \in \Psi_{\mathrm{sc}}^{-1,0}$ for $F > 0$. In addition, if we choose $\chi \in C_c^\infty(\mathbb{R})$ appropriately with $\chi \geq 0, \chi(0) = 1$, its principal symbol, including the boundary symbol, is elliptic.*

Proof. For the original derivation of the decay property of A_F's Schwartz kernel and consequently the membership $A_F \in \Psi_{\mathrm{sc}}^{-1,0}$, we refer readers to [11, Section 3.5]. And see [9, Section 5] for the interior of the region.

Since we only concern principal symbol level information, we use the following quantization formula:

$$q_L(a)u(x,y) = (2\pi)^{-n} \int e^{i(\xi \frac{x-x'}{x^2} + \eta \frac{y-y'}{x})} u(x',y') a(x,y,\xi,\eta) \frac{dx'dy'}{(x')^{n+1}} d\xi d\eta. \tag{8.9}$$

which is more convenient to evaluate in the current setting and does not differ from (7.9) on the principal symbol level, to compute the principal symbol. The Schwartz kernel of A_F is given by:

$$K_{A_F}(z,z') = (2\pi)^{-2} \int e^{i(z-z')\cdot\zeta} a_F(z,\zeta) d\zeta,$$

where a_F is the left symbol of A_F. From the definition of A_F, we know

$$\begin{aligned} K_{A_F}(z,z') = &\int e^{-\frac{F}{x} + \frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} x^{-2} \chi(\frac{\lambda}{x}, y) \delta(z' - \gamma_{x,y,\lambda,\omega}(t)) \\ &dt|d\nu| \\ = (2\pi)^{-n} &\int e^{-\frac{F}{x} + \frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} x^{-2} \chi(\frac{\lambda}{x}, y) e^{-i\zeta' \cdot (z' - \gamma_{x,y,\lambda,\omega}(t))} \\ &dt|d\nu||d\zeta'|. \end{aligned}$$

Taking inverse Fourier transform in z' and then evaluate at ζ, which turns out to be a factor $\delta_0(\zeta - \zeta')$, we know:

$$\begin{aligned} a_F(z,\zeta) = &(2\pi)^n e^{-iz\cdot\zeta} \mathcal{F}^{-1}_{z'\to\zeta} K_{A_F}(z,z') \\ = &\int e^{-\frac{F}{x} + \frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} x^{-2} \chi(\frac{\lambda}{x}, y) e^{-iz\cdot\zeta} e^{i\zeta\cdot\gamma_{x,y,\lambda,\omega}(t)} \\ &dt|d\nu|. \end{aligned}$$

Suppose we use the coordinate in the scattering cotangent vectors: $\zeta = \xi \frac{dx}{x^2} + \eta \frac{dy}{x}$ and use the coordinate $\gamma_{x,y,\lambda,\omega}(t) = (\gamma^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t))$. In our context, all of them are scalars. While in the general $n-$dimensional case, the first component is of dimension one, and the second component has dimension $n-1$. The expression above becomes

$$a_F(z,\zeta) = \int e^{-\frac{F}{x}+\frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} x^{-2}\chi(\frac{\lambda}{x}, y) e^{i(\frac{\xi}{x^2},\frac{\eta}{x})\cdot(\gamma^{(1)}_{x,y,\lambda,\omega}(t)-x,\gamma^{(2)}_{x,y,\lambda,\omega}(t)-y)} dt|d\nu|. \tag{8.10}$$

Next we investigate the phase function of this oscillatory integral and then apply the stationary phase lemma. We denote components of $\gamma_{z,\nu}(t)$ (recall $\nu = (\lambda,\omega)$) and its derivatives by

$$\begin{aligned} &\gamma_{x,y,\lambda,\omega}(0) = (x,y), \quad \dot{\gamma}_{x,y,\lambda,\omega}(0) = (\lambda,\omega), \\ &\ddot{\gamma}_{x,y,\lambda,\omega}(t) = 2(\alpha(x,y,\lambda,\omega,t), \beta(x,y,\lambda,\omega,t)), \end{aligned} \tag{8.11}$$

where α, β are defined by by this equation and are smooth with respect to their variables. In addition, α is a quadratic form in ω, and it is strictly positive definition in ω for small enough x, λ, t, which means being close to the starting point at the boundary, by our convexity condition.

Since $\gamma_{x,y,\lambda,\omega}$ starting at (x,y) with initial velocity (λ,ω), there exists smooth functions $\Gamma^{(1)}, \Gamma^{(2)}$ such that

$$\gamma_{x,y,\lambda,\omega}(t) = (x + \lambda t + \alpha t^2 + \Gamma^{(1)}(x,y,\lambda,\omega,t)t^3, y + \omega t + \Gamma^{(2)}(x,y,\lambda,\omega,t)t^2),$$

where we only expand the second component to the first order and have included the $\beta-$term in the definition of $\Gamma^{(2)}$. Then we make the change of variables

$$\hat{t} = \frac{t}{x}, \quad \hat{\lambda} = \frac{\lambda}{x}.$$

By the support condition of χ, the integrand is none-zero when $\hat{\lambda}$ is in a compact interval. However, the bound on $\hat{t}$ is $|\hat{t}| \leq \frac{T_g}{x}$, which is not uniformly bounded, we amend this by treating it in two regions separately. Using these new variables, we rewrite our phase as

$$\phi = \xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + x\hat{t}^3\Gamma^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t})) + \eta(\omega\hat{t} + x\hat{t}^2\Gamma^{(2)}(x,y,x\hat{\lambda},\omega,x\hat{t})).$$

The damping factor coming from exponential conjugation is

$$\begin{aligned} -\frac{F}{x} + \frac{F}{\gamma^{(1)}_{x,y,\lambda,\omega}(t)} =& -F(\lambda t + \alpha t^2 + t^3\Gamma^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t})) \\ &\times (x(x + \lambda t + \alpha t^2 + t^3\Gamma^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t})))^{-1} \\ =& -F(\hat{\lambda}\hat{t} + \alpha\hat{t}^2 + \hat{t}^3 x\hat{\Gamma}^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t})), \end{aligned}$$

where $\hat{\Gamma}^{(i)}$ is introduced when we first express $\gamma^{(1)}_{x,y,\lambda,\omega}(t)$ by variables t, λ, and then invoke our change of variables, then collect the remaining terms, which is a smooth function of these normalized variables. So this amplitude is Schwartz in $\hat{t}$, hence we take a constant $\epsilon_t > 0$ and deal with regions $|\hat{t}| \geq \epsilon_t$ and $|\hat{t}| < \epsilon_t$ separately. In our later argument, we will take ϵ_t small to enforce $\hat{t} = 0$ holds for critical points.

To see the peudodifferential property, we analyze the damping factor by analyzing the structure of geodesics. By the convexisty condition (8.2) we assumed, there exists a $\lambda_0 > 0$ and $C > 0$ such that for all $\lambda < \lambda_0$, we have (whenever $\gamma_{x,y,\lambda,\omega}(t)$ is defined):

$$\gamma^{(1)}_{x,y,\lambda,\omega}(t) \geq x + \lambda t + Ct^2. \tag{8.12}$$

Then by the same computation above, and notice that $x\hat{t}^3 = t \times \hat{t}^2$, which is $\ll \hat{t}^2$ if we select c_0 to be small so that the uniform upper bound of t is small, we know that this phase is $O(e^{-C\hat{t}^2})$ for large $\hat{t}$ with some $C > 0$. In particular, those stationary phase results for compact intervals applies here as well.

We left the proof of those geometric facts in Exercse 3. They were proven in [14, Lemma 3.1].

To summarize, after substituting ϕ and the damping factor above into (8.10), it is not hard to see, for finite (ξ, η) that it remain bounded under repeated application of $x\partial_x$ and ∂_y. And indeed, the $\chi(\lambda/x)$-factor tell us that we might lose control if we consider the classical pseudodifferential algebra with respect to (x, y)[13] here.

In addition, the Gaussian in $\hat{t}$ decay allows the stationary phase argument below down to arbitrary many terms, and this also shows the symbolic behaviour as $(\xi, \eta) \to \infty$. See Exercise 4.

Before considering the critical points of the phase for small $x > 0$, which is what we finally need, we first consider the critical points of the phase at $x = 0$. This helps us to get rid of those $\Gamma^{(i)}-$terms and simplifies the process to solve the equation for the critical points.

When $x = 0$, the phase becomes

$$\xi(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \hat{t}\eta \cdot \omega.$$

When $|\hat{t}| \geq \epsilon_t$, the derivative with respect to $\hat{\lambda}$ vanishes only when $\xi = 0$. Since our analysis on the ellipticity is happening away from the zero section, thus $\xi = 0$ implies $|\eta| \gtrsim 1$. Then we consider the ω-derivative to see that there is no critical points here. So the region $|\hat{t}| \geq \epsilon_t$ gives rapid decay contribution when $x = 0$.

The case $x > 0$ can be dealt with the same method, but with more complicated computation. Notice that, $\alpha, \Gamma^{(i)}$ take $\lambda = x\hat{\lambda}, t = x\hat{t}$ as variables, and produces an extra x factor when we take partial derivatives with respect to $\hat{\lambda}, \hat{t}$. Concretely, the derivative with respect to $\hat{\lambda}$ is:

$$\begin{aligned}\frac{\partial \phi}{\partial \hat{\lambda}} &= \xi\hat{t}(1 + x\hat{t}\partial_\lambda\alpha + x^2\hat{t}^2\partial_\lambda\Gamma^{(1)}) + \eta x^2\hat{t}^2\partial_\lambda\Gamma^{(2)} \\ &= \xi\hat{t}(1 + t\partial_\lambda\alpha + t^2\partial_\lambda\Gamma^{(1)}) + \eta t^2\partial_\lambda\Gamma^{(2)}.\end{aligned}$$

Recall that $|t| \leq T_g$ and we can choose T_g to be small by shrinking O_p. Thus when both $|\xi| \geq C|\eta|$ and $|\hat{t}| \geq \epsilon_t$ hold, $\xi\hat{t}$ is non-zero and is going to dominate other terms, so $\frac{\partial \phi}{\partial \hat{\lambda}}$ can not vanish and there is no critical point in this case.

In fact, expression for $\frac{\partial \phi}{\partial \hat{\lambda}}$ above shows that we have to have $\frac{|\xi|}{|\eta|}\hat{t} \ll 1$ at critical points (if we still only consider $|\hat{t}| \geq \epsilon_t$). Then we decompose ω according to the direction of

[13]Be careful here! We would be at ‘finite place’ if we use the classical pseudodifferential algebra.

η to write

$$\eta \cdot \omega = |\eta|\omega^{\|}, \ \omega = \omega^{\|} + \omega^{\perp}. \tag{8.13}$$

Then we have

$$\phi = |\eta|\big(\frac{\xi}{|\eta|}(\hat{\lambda}\hat{t} + \alpha\hat{t}^2) + \hat{t}\omega^{\|}\big).$$

Setting the $\hat{t}$-derivative to 0 and recall $\frac{|\xi|}{|\eta|}\hat{t} \ll 1$, we know $|\omega^{\|}| \ll 1$ and it is a valid coordinate component for ω. Finally taking $\omega^{\|}$-derivative gives $\hat{t} = 0$, contradiction (or come to the next case).

Next we consider the region $|\hat{t}| < \epsilon_t$, whose closure is compact, and consequently we can apply the stationary phase lemma. The same as before, we consider the condition that the derivative with respect to $\hat{\lambda}$ and $\hat{t}$ vanish. First consider the $x = 0$, in which case the expression can be significantly simplified:

$$\xi\hat{t} = 0, \quad \xi\hat{\lambda} + \eta \cdot \omega = 0.$$

Repeating the argument before, we have the condition for critical points:

$$\hat{t} = 0, \quad \xi\hat{\lambda} + \eta \cdot \omega = 0.$$

Further, since the ϵ_t in arguments above is arbitrary, we know that the condition $\hat{t} = 0$ holds for any critical point including the $x \neq 0$ case. The second condition can be derived if we notice that (for general x):

$$\frac{\partial\phi}{\partial\hat{t}} = (\xi\hat{\lambda} + \eta \cdot \omega) + O(\hat{t}),$$

where the $O(\hat{t})$ term vanishes when $\hat{t} = 0$, and can be computed explicitly:

$$(2\xi\alpha + 2x\Gamma^{(2)})\hat{t} + (3x\xi\Gamma^{(1)} + x^2\partial_t\Gamma^{(2)})\hat{t}^2 + \xi x^2\partial_t\Gamma^{(1)}\hat{t}^3.$$

So those two conditions for stationary points extends to the $x \neq 0$ case. In order to apply those conditions of critical points of the phase, we first rewrite (8.10) as:

$$\begin{aligned} a_F(z,\zeta) &= \int e^{-\frac{F}{x} + \frac{F}{x(\gamma_{x,y,\lambda,\omega}(t))}} x^{-2}\chi(\hat{\lambda}) e^{i(\frac{\xi}{x^2},\frac{\eta}{x})\cdot(\gamma^{(1)}_{x,y,\lambda,\omega}(t)-x,\gamma^{(2)}_{x,y,\lambda,\omega}(t)-y)} dt|d\nu| \\ &= \int e^{-F(\hat{\lambda}\hat{t}+\alpha\hat{t}^2+\hat{t}^3x\hat{\Gamma}^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t}))}\chi(\hat{\lambda}) \\ &\quad e^{i(\xi(\hat{\lambda}\hat{t}+\alpha\hat{t}^2+\hat{t}^3x\hat{\Gamma}^{(1)}(x,y,x\hat{\lambda},\omega,x\hat{t}))+\eta(\omega\hat{t}+x\hat{t}^2\Gamma^{(2)}(x,y,x\hat{\lambda},\omega,x\hat{t})))} d\hat{t}d\hat{\lambda}d\omega. \end{aligned} \tag{8.14}$$

We use the notation $\theta = (\hat{\lambda}, \omega)$ and apply the stationary phase lemma with respect to $\hat{t}, \theta$ to compute the leading part as $|(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})| \to \infty$. We decompose θ according to directions parallel to and orthogonal to $(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})$ and denote projections of θ by $\theta^{\|}, \theta^{\perp}$ respectively. Then the critical set is given by $\hat{t} = 0, \theta^{\|} = 0$. So the leading part, up to a constant factor, is

$$|(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})|^{-1} x \int_{\mathbb{S}^{n-2}} \chi(\hat{\lambda}(\theta^{\perp}), \omega(\theta^{\perp}))d\theta^{\perp}, \tag{8.15}$$

where the $|(\xi_{\mathrm{sc}}, \eta_{\mathrm{sc}})|^{-1}$ comes from the square root of the determinant of the Hessian of the phase in the stationary phase lemma and $\hat{\lambda}(\theta^{\perp}), \omega(\theta^{\perp})$ indicates that this critical set is parametrized by $\theta^{\perp}$ and thus other variables are functions of it. Now if we choose

$\chi \geq 0$ with $\chi(0,y) = 1$, then this indeed gives an elliptic (in the differential sense) symbol.

Next we turn to show boundary part of the principal symbol of A_F is also elliptic (when the fiber variables are finite). Evaluating (8.14) at $x=0$, the boundary principal symbol of A_F is

$$a_F(0,y,\zeta) = \int e^{-F(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)}\chi(\hat{\lambda},y)e^{i(\xi(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)+\hat{t}\eta\cdot\omega)}d\hat{t}d\hat{\lambda}d\omega.$$

Now $\alpha(x,y,x\hat{\lambda},\omega) = \alpha(0,y,0,\omega) := \alpha(y,\omega)$, which is a positive quadratic form in ω, hence changing the sign of ω does not change its value. Now an observation is that we can allow χ to depend on y and denote it by $\chi(s,y)$. We choose them to be a Gaussian density first, then we use approximation argument to obtain one that has compact support in s. We choose $\chi(s,y) = e^{-\frac{Fs^2}{2\alpha(y)}}$, then we have:

$$\begin{aligned}&\int e^{-F(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)}\chi(\hat{\lambda})e^{i(\xi(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)+\omega\cdot\eta\hat{t})}d\hat{t}d\hat{\lambda}d\omega\\ =&\int(\int e^{-F\hat{\lambda}\hat{t}-\frac{F\hat{\lambda}^2}{2\alpha}+i\xi\hat{\lambda}\hat{t}}d\hat{\lambda})e^{-F\alpha\hat{t}^2+i\omega\cdot\eta\hat{t}+i\xi\alpha\hat{t}^2}d\hat{t}d\omega\end{aligned}$$

The integral in $\hat{\lambda}$ is a Fourier transform of Gaussian density, it is $\sqrt{\frac{2\pi\alpha}{F}}e^{\frac{\alpha F\hat{t}^2}{2}-i\xi\alpha\hat{t}^2-\frac{\alpha}{2F}\hat{t}^2\xi^2}$. Thus we need to compute:

$$\int e^{-\frac{\alpha}{2F}(F^2+\xi^2)\hat{t}^2+i\omega\cdot\eta\hat{t}}d\hat{t},$$

which is again a Gaussian type integral, and it equals to a constant multiple of

$$\sqrt{\frac{F}{\alpha}}(F^2+\xi^2)^{-\frac{1}{2}}e^{-\frac{F(\omega\cdot\eta)^2}{2(\xi^2+F^2)\alpha(y,\omega)}},$$

which is even in η. Finally, with a constant factor $\hat{C}$, we have:

$$a_F(0,y,\zeta) = \hat{C}\int_{\mathbb{S}^{n-2}}\sqrt{\frac{F}{\alpha}}(F^2+\xi^2)^{-\frac{1}{2}}e^{-\frac{F(\omega\cdot\eta)^2}{2(\xi^2+F^2)\alpha(y,\omega)}}d\omega. \tag{8.16}$$

When $x=0$, this equation shows that $a_F(0,y,\zeta)$ is lower bounded by a positive constant when the fiber variables are uniformly bounded. In fact, a more detailed decomposition in ω also gives lower bound as $|(\xi,\eta)| \to \infty$. The only potential issue to make it not lower bounded is when $|\omega\cdot\eta| \gg |\xi|^2 \gg F^2$. But one can look at the part on $\mathbb{S}^{n-2}$ that is perpendicular to η with radius $\sim \frac{(|\xi|^2+F^2)^{1/2}}{|\eta|}$. This part is roughly a small interval times $\mathbb{S}^{n-3}$ and will have volume $\sim \frac{(|\xi|^2+F^2)^{1/2}}{|\eta|}$ and the factor $e^{-\frac{F(\omega\cdot\eta)^2}{2(\xi^2+F^2)\alpha(y,\omega)}}$ is uniformly lower bounded here. So the integral is $\gtrsim 1$ even as $|(\xi,\eta)| \to \infty$, which is how it should be, according to the previous part about ellipticity near fiber infinity.

Notice that the constants in (8.16) are uniformly bounded for all choices of small c, thus we can choose c small and it will be lower bounded by a positive constant for $0 \leq x \leq c$ by continuity (which is uniform in terms of choices of c).

Now we amend the compact support issue. Let χ be a Gaussian as above, which generates an elliptic operator, then we pick a sequence $\chi_n \in C_c^\infty(\mathbb{R})$ converges toχ in the Schwartz function space $\mathcal{S}(\mathbb{R})$. Then we can obtain the convergence of $\hat{\chi}_n$ to $\hat{\chi}$

in Schwartz function space. This gives us the convergence of $X-$Forier transform in $\mathcal{S}(\mathbb{R})$. The $Y-$Fourier transform step is also continuous with respect to the topology of $\mathcal{S}(\mathbb{R})$. In particular, we obtain the convergence of $|\zeta|a_{n,F}(z,\zeta)$, the symbol obtained from χ_n, in the C^0 topology, which is enough to derive an elliptic type estimate for χ_n with large enough n. $\square$

Remark 8.2. Looking at (8.10),

8.5. **The proof of the main theorem.** Fix c_0 small and apply results in previous sections to Ω_{c_0}, estimates above are uniform with respect to $c \in (0, c_0]$. We let c vary and take $f \in H^s_F(O_p)$. For Ω_c, denote A_F in Section 8.4 constructed for Ω_c by B_c. By Proposition 8.2, By the ellipticity of B_c we have its parametrix G_c such that $G_cB_c = \mathrm{Id} + E_{0c}$, $E_{0c} \in \Psi^{-\infty,-\infty}_{\mathrm{sc}}(\tilde{X})$.

Consider the map $\Psi_c(\tilde{x}, y) = (\tilde{x} + c, y)$, and let

$$A_c = (\Psi_c^{-1})^* B_c \Psi_c^*,\ E_c = (\Psi_c^{-1})^*(E_{0c})\Psi_c^*. \tag{8.17}$$

This conjugation is introduced to make this family of operators to be defined on a fixed region $\bar{M}_0 := \{\tilde{x} \geq 0\}$. We have an estimate of the error term in terms of f. To be more precise, we consider the Schwartz kernel K_{E_c} of E_c, which satisfies $|x^{-N}x'^{-N}K_{E_c}| \leq C_N$ on Ω_c. Then we insert a truncation factor ϕ_c compactly supported in Ω_c, and being identically 1 on smaller compact set K_c, such that $|\phi_c(x,y)\phi_c(x',y')K_{E_c}| \leq C'_N c^{2N} x^{n+1}(x')^{n+1}$ for all N. The $(n+1)$-power factors are introduced to deal with the scattering density. Then we apply Schur's lemma on the integral operator bound (together with the aforementioned $N-th$ power estimate) to conclude that

$$||\phi_c E_c \phi_c||_{L^2_{\mathrm{sc}}(\bar{M}_0)\to L^2_{\mathrm{sc}}(\bar{M}_0)} \leq C''_N c^{2N}. \tag{8.18}$$

In particular, we can take c_0 so that this norm < 1 when $c \in (0, c_0]$. Since those conjugations are invertible, this guarantees that $\phi_c G_c B_c \phi_c = \mathrm{Id} + \phi_c E_{0c} \phi_c$ is invertible. So for the functions supported on K_c, B_c is injective. K_c can be arbitrary compact subset of Ω_c for arguments up to now. Support conditions are encoded by subscripts below. For example, $H^{s,r}_{\mathrm{sc}}(\bar{M}_c)_{K_c}$ is the space consists of those functions in $H^{s,r}_{\mathrm{sc}}(\bar{M}_c)$ which have support in K_c. Define $\bar{M}_c := \{\tilde{x} + c \geq 0\}$ and $K_c := \bar{M}_c \cap \{\rho \geq 0\} = \Omega_c$. K_c is compact by our choice of $\tilde{x}$. We have

$$||v||_{H^{s,r}_{\mathrm{sc}}(\bar{M}_c)_{K_c}} \leq C||B_c v||_{H^{s+1,r}_{\mathrm{sc}}(\bar{M}_c)}.$$

If we recover this expression to A, this is (with $f = e^{\frac{F}{x}}v$):

$$||f||_{e^{\frac{F}{x}}H^{s,r}_{\mathrm{sc}}(\bar{M}_c)_{K_c}} \leq C||Af||_{e^{\frac{F}{x}}H^{s+1,r+1}_{\mathrm{sc}}(\bar{M}_c)}.$$

We can get rid of the $r-$indices with the cost of increasing the power of left hand side to $e^{\frac{F+\delta}{x}}$. That is:

$$||f||_{e^{\frac{F+\delta}{x}}H^{s}_{\mathrm{sc}}(\bar{M}_c)_{K_c}} \leq C||Af||_{e^{\frac{F}{x}}H^{s+1}_{\mathrm{sc}}(\bar{M}_c)}.$$

Finally we consider the boundedness of operators involved. We consider the decomposition $A = L \circ I_0$, and show that L is bounded. In order to prove this, we decompose

L into $L = M_2 \circ \Pi \circ M_1$, with M_2, Π, M_1 being

$$
\begin{aligned}
M_1 :& H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y \times \mathbb{R}_\lambda \times \mathbb{S}^{n-2}_\omega) \to H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y \times \mathbb{R}_\lambda \times \mathbb{S}^{n-2}_\omega),\\
&(M_1 u)(x,y,\lambda,\omega) = x^s \chi(\frac{\lambda}{x}, y) u(x,y,\lambda,\omega),\\
\Pi :& H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y \times \mathbb{R}_\lambda) \to H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y), \quad (\Pi u)(x,y) = \int_{\mathbb{R}} u(x,y,\lambda,1) d\lambda,\\
M_2 :& H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y) \to x^{-(s+1)} H^s([0,+\infty)_x \times \mathbb{R}^{n-1}_y), \quad (M_2 f)(x,y) = x^{-(s+1)} f(x,y).
\end{aligned}
$$

Consider the boundedness of M_1 when $s \in \mathbb{N}$ first. The general case follows from interpolation. Consider derivatives of $x^s \chi(\frac{\lambda}{x}, y) u(x,y,\lambda,\omega)$ up to order s. Each order of differentiation on χ gives an x^{-1} factor, which is canceled by x^s and the remaining part belongs to L^2 by smoothness of χ and $u \in H^s$. M_2 is bounded by the definition of the space on the right hand side. The operator Π is a pushforward map, integrating over $|\lambda| \le C|x|$ (notice the support condition after we apply M_1), hence bounded via Minkowski inequality.

On the other hand, I_0 itself is a bounded operator. This comes from the decomposition $I_0 = \tilde{\Pi} \circ \Phi^*$, where Φ is the geodesic coordinate representation $\Phi(z,\nu,t) = \gamma_{z,\nu}(t)$ and $\tilde{\Pi}$ is integrating against t, which is bounded as a pushforward map. Because the initial vector always has length 1 on the tangent component, the travel time is uniformly bounded. Φ is one component of Γ and the later is a diffeomorphism when we shrink the region. So Φ has surjective differential, hence the pull back is bounded. Consequently I_0 is bounded.

The boundedness of L gives us an estimate

$$||Af||_{e^{\frac{F}{x}} H^{s+1}_{\mathrm{sc}}(\bar{M}_c)} \le C_1 ||I_0 f||_{H^{s+1}(PSX|_{\bar{M}_c})},$$

where we require f to have supported in K_c, and used the fact $\mathbb{R}_\lambda \times \mathbb{S}^{n-2}_\omega$ parametrizes $\mathbb{S}^{n-1}$ apart from two poles, and this completes the proof.

Exercises.

(1) Verify that $\tilde{x}$ given by (8.5) has properties we want.
(2) What is the expression of the damping factor introduced by the conjugation in terms of $X = \frac{x-x'}{xx'}$? Why this is good for us?
(3) Prove (8.12), and verify the claim about the uniform upper bound of t.
(4) Prove that a_F in (8.10) is indeed symbolic with respect to scattering frequencies (ξ, η). (Hint: Fix the order of derivatives you want to deal with, then perform stationary expansion up to some level depending on this order.)
(5) (i)Why this proof does not work in two dimension?
(ii) What conclusion can we say in two dimension using this proof?

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