Yang-Baxter Equations and Clifford Algebras

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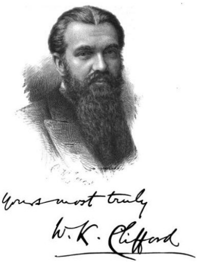
Baxter2025 : Exactly Solved Models and Beyond
Celebrating the life and achievements of Rodney James Baxter

Clifford Algebras



William Kingdon Clifford (1845-1879)

A Clifford algebra $\mathbf{CL}(p,q)$ of order $p+q$ is an associative algebra generated by $\{\Gamma_1, \cdots, \Gamma_{p+q}\}$ satisfying,

$$\begin{aligned}
\Gamma_i^2 &= \mathbb{1} \text{ for } 1 \leq i \leq p, \\
\Gamma_i^2 &= -\mathbb{1} \text{ for } p+1 \leq i \leq p+q, \\
\Gamma_i \Gamma_j &= -\Gamma_j \Gamma_i \text{ for } i \neq j.
\end{aligned}$$

Setup

- Pick a pair of anticommuting operators A, B:

$$AB = -BA.$$

- No further restrictions.
- They can be realized as product of the Clifford generators Γ.
- Let them act on vector spaces [local Hilbert space, V]:

$$\{A_i, B_i\} = 0$$
$$[A_i, B_j] = 0 \; ; \; i \neq j.$$

The indices i, j denote the respective copies of V in the tensor product.

- These operators form the algebraic input for our ansätze.
- We solve n-Simplex Equations, where n is the spacetime dimension.

$n = 2$ - **Simplex Equation**
- **Yang-Baxter Equation** -

Ansatz

- Consider the ansätze :

$$R_{ij} = A_i A_j \quad ; \quad R_{ij} = B_i B_j.$$

- They trivially solve the non-braided YBE :

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

- Reason : Each index appears twice in the YBE.

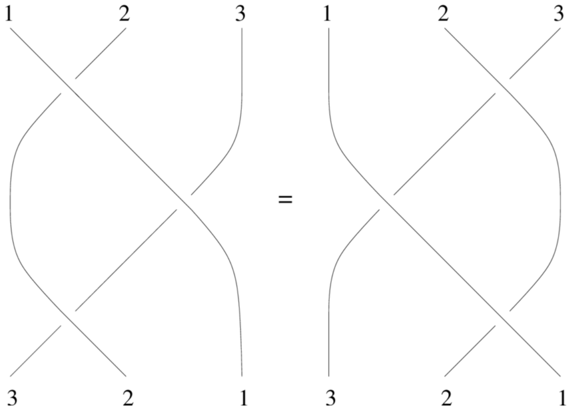
$$A_1^2 A_2^2 A_3^2 = A_1^2 A_2^2 A_3^2$$

- Note : The braid operator $\check{R}_{ij} = P_{ij} R_{ij}$ solves the braided YBE non-trivially :

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23}.$$

P_{ij} is the standard permutation operator on the tensor products of V's.

Pictorially



Linearity

- Question - Do linear combinations of A_iA_j and B_iB_j satisfy the non-braided YBE ?
- Let $R_{ij} = A_iA_j + B_iB_j$

$$
\begin{aligned}
& R_{12}R_{13}R_{23} \\
= \quad & A_2A_3\left(A_1A_2 - B_1B_2\right)\left(A_1A_3 - B_1B_3\right) \\
+ \quad & B_2B_3\left(-A_1A_2 + B_1B_2\right)\left(-A_1A_3 + B_1B_3\right) \\
= \quad & R_{23}\left(A_1A_2 - B_1B_2\right)\left(A_1A_3 - B_1B_3\right) \\
= \quad & R_{23}\left[A_1A_3\left(A_1A_2 + B_1B_2\right)\right. \\
- \quad & \left. B_1B_3\left(-A_1A_2 - B_1B_2\right)\right] \\
= \quad & R_{23}R_{13}R_{12}
\end{aligned}
$$

- Linear solutions for non-linear equations

$n = 3$ - **Simplex Equation**
- **Tetrahedron Equation** -

Labeling Schemes

- Different ways to index the scattering process.
- Vertex Form : Labels the vertices at the intersections.

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$

- Edge Form : Labels the line segments.

$$R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123}.$$

- Cell Form : Labels the tetrahedrons formed in spacetime picture. Equation satisfied by the Boltzmann Weights. [Zamolodchikov '80, '81].

Ansätze

- Consider a type (a, b) operator.

$$a = \text{Number of } A\text{'s} \quad : \quad b = \text{Number of } B\text{'s}.$$

- Solutions are 'words' made of four types of operators [8 words]:

$$\begin{aligned}
(3,0) \quad &: \quad R_{ijk} = A_i A_j A_k \\
(2,1) \quad &: \quad R_{ijk} = \{A_i A_j B_k, \text{or } A_i B_j A_k, \text{or } B_i A_j A_k\}, \\
(1,2) \quad &: \quad R_{ijk} = \{B_i B_j A_k, \text{or } B_i A_j B_k, \text{or } A_i B_j B_k\}, \\
(0,3) \quad &: \quad R_{ijk} = B_i B_j B_k,
\end{aligned}$$

satisfy vertex and edge forms of the tetrahedron equation.

- Reason : Both sides of this equation simplify to

$$A_1^2 A_2^2 (B_3 A_3) A_4^2 (B_5 A_5) A_6^2,$$

when $R_{ijk} = A_i A_j B_k$ is used.

Linear space

• Linear combinations of operators of types $(2, 1)$ and $(0, 3)$ satisfy the vertex and edge forms.

$$R_{ijk} = \alpha\ A_i A_j B_k + \beta\ A_i B_j A_k + \gamma\ B_i A_j A_k + \delta\ B_i B_j B_k .$$

• Linear combinations of operators of types $(1, 2)$ and $(3, 0)$ satisfy the vertex and edge forms.

$$R_{ijk} = \alpha\ B_i B_j A_k + \beta\ B_i A_j B_k + \gamma\ A_i B_j B_k + \delta\ A_i A_j A_k .$$

• The parameters α, β, γ and δ are constant complex numbers, like coupling constants.

Spectral parameter dependent solutions

- Make the coefficients site-dependent :

$$R_{ijk}(\Sigma_{ijk}) = \alpha_{ijk}\ A_iA_jB_k + \beta_{ijk}\ A_iB_jA_k + \gamma_{ijk}\ B_iA_jA_k + \delta_{ijk}\ B_iB_jB_k.$$

Here $\Sigma_{ijk} = (\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}, \delta_{ijk})$ denotes the tuple of spectral parameters.

- This satisfies spectral-parameter dependent tetrahedron equation:

$$\begin{aligned} & R_{123}(\Sigma_{123})R_{145}(\Sigma_{145})R_{246}(\Sigma_{246})R_{356}(\Sigma_{356}) \\ = & R_{356}(\Sigma_{356})R_{246}(\Sigma_{246})R_{145}(\Sigma_{145})R_{123}(\Sigma_{123}). \end{aligned}$$

$n = 4$ - **Simplex Equation**

[V. Bazhanov, Y.G. Stroganov (1982)]

Ansätze

- The vertex form of the 4-simplex equation :

$$
\begin{aligned}
& R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} \\
= \quad & R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}.
\end{aligned}
$$

- 'Words' in A and B [5 types of operators] :

$$
\begin{aligned}
(4,0) \quad &: \quad R_{ijkl} = A_iA_jA_kA_l, \\
(3,1) \quad &: \quad R_{ijkl} = \{A_iA_jA_kB_l, A_iA_jB_kA_l, A_iB_jA_kA_l, B_iA_jA_kA_l\}, \\
(2,2) \quad &: \quad R_{ijkl} = \{A_iA_jB_kB_l, A_iB_jA_kB_l, B_iA_jA_kB_l \\
& \qquad A_iB_jB_kA_l, B_iA_jB_kA_l, B_iB_jA_kA_l\}, \\
(1,3) \quad &: \quad R_{ijkl} = \{B_iB_jB_kA_l, B_iB_jA_kB_l, B_iA_jB_kB_l, A_iB_jB_kB_l\}, \\
(0,4) \quad &: \quad R_{ijkl} = B_iB_jB_kB_l.
\end{aligned}
$$

Linearity in the space of solutions

• Other than the $(3,1)$ and $(1,3)$ types all other words satisfy the 4-simplex equation.

• Most general solution is a linear combination of $(4,0)$, $(0,4)$ and $(2,2)$ type words :

$$\begin{aligned} R_{ijkl} &= \alpha\ A_iA_jA_kA_l + \gamma\ B_iB_jB_kB_l \\ &+ \beta_1\ A_iA_jB_kB_l + \beta_2\ A_iB_jA_kB_l + \beta_3\ B_iA_jA_kB_l \\ &+ \beta_4\ A_iB_jB_kA_l + \beta_5\ B_iA_jB_kA_l + \beta_6\ B_iB_jA_kA_l. \end{aligned}$$

• The other words satisfy generalizations of the 4-simplex equation.

• Make the coefficients site-dependent for spectral parameter dependent solutions.

General n-Simplex Equation

Remarks on n-Simplex Operators

- n indices on the n-simplex operator.
- An (a, b) type word

$$A_{i_1} \cdots A_{i_a} B_{j_1} \cdots B_{j_b} \ ; \ i_1, \cdots j_b \in \{1, \cdots n\}.$$

$$a + b = n.$$

- There are $n + 1$ types of such words.
- The vertex form of the n-simplex equation has $\frac{n(n+1)}{2}$ indices.

Theorem for n-simplex solutions

• This theorem specifies when words are solutions and the conditions for linearity in the space of the different types of words.

Condition on words to be solutions : *Consider the set of type* (a, b) *words where* $a, b \in \{0, 1, \cdots, n\}$ *and* $a + b = n$*. These words satisfy the n-simplex equation when at least one of a or b is even.*

Linearity within a given type : *Linear combination of the words of a given type* (a, b) *is also a solution.*

Linearity between different types : *Linear combinations of different pairs are solutions when the pairs* (a_i, b_i)*, with* $i \in \{1, \cdots, n+1\}$*, are such that* $a_i - a_j$ *is even for all pairs* $i, j \in \{1, \cdots, n+1\}$*.*

Anti-n-Simplex Equation

n is even

- Linear combinations of the type (a, b) operators when both a and b are odd satisfy the anti-d-simplex equation.

- Example $n = 2$: The $(1, 1)$ type operators

$$R_{ij} = \alpha \; A_i B_j + \beta \; B_i A_j$$

satisfy the anti-Yang-Baxter equation

$$R_{12} R_{13} R_{23} = - \; R_{23} R_{13} R_{12}.$$

- Example $n = 4$: Linear combinations of the $(3, 1)$ and $(1, 3)$ types satisfy the anti-4-simplex equation :

$$\begin{aligned} & R_{1234} R_{1567} R_{2589} R_{368,10} R_{479,10} \\ = & \; - \; R_{479,10} R_{368,10} R_{2589} R_{1567} R_{1234}. \end{aligned}$$

n is odd

- Linear combinations of (a_1, b_1) and (a_2, b_2) types when $a_1 - a_2$ is odd.

- Example $n = 3$: Linear combinations of types $(3, 0)$ and $(0, 3)$ satisfy the anti-tetrahedron identity

$$R_{123}R_{145}R_{246}R_{356} = R^{(-)}_{356}R^{(-)}_{246}R^{(-)}_{145}R^{(-)}_{123}.$$

- Here

$$\begin{aligned} R_{ijk} &= \alpha\ A_iA_jA_k + \beta\ B_iB_jB_k. \\ R^{(-)}_{ijk} &= \alpha\ A_iA_jA_k - \beta\ B_iB_jB_k. \end{aligned}$$

Reflection Equation

[E. K. Sklyanin (1988), I. V. Cherednik (1984)]

[A. Kuniba - Quantum Groups in Three Dimensional Integrability (2022), Springer]

$n = 2, 3$

- $n = 2$ case :

$$R_{12}K_2R_{21}K_1 = K_1R_{12}K_2R_{21},$$

solved by

$$R_{ij} = A_iA_j + B_iB_j \;;\; K_j = A_j + B_j.$$

- $n = 3$ case :

$$\begin{aligned} R_{ijk} &= A_iA_jB_k + A_iB_jA_k + B_iA_jA_k \\ K_{ijkl} &= A_iA_jA_kA_l + B_iB_jB_kB_l. \end{aligned}$$

solves

$$\begin{aligned} & R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} \\ = \;& R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}. \end{aligned}$$

Examples : Matrix Solutions

Take $V = \mathbb{C}^2$.

X, Y and Z are the Pauli matrices.

Yang-Baxter Solutions

- Choose $A = X$ and $B = Z$.

$$R(\mu_1, \mu_2) = \mu_1 \; X \otimes X + \mu_2 \; Z \otimes Z = \begin{pmatrix} \mu_2 & \cdot & \cdot & \mu_1 \\ \cdot & -\mu_2 & \mu_1 & \cdot \\ \cdot & \mu_1 & -\mu_2 & \cdot \\ \mu_1 & \cdot & \cdot & \mu_2 \end{pmatrix}.$$

- Choose $A = X\left(\frac{\mathbb{1}+Z}{2}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $B = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$R(\mu_1, \mu_2) = \mu_1 \; A \otimes A + \mu_2 \; Z \otimes Z = \begin{pmatrix} \mu_2 & \cdot & \cdot & \cdot \\ \cdot & -\mu_2 & \cdot & \cdot \\ \cdot & \cdot & -\mu_2 & \cdot \\ \mu_1 & \cdot & \cdot & \mu_2 \end{pmatrix}$$

- Coincides with the $(1, 4)$ and $(0, 1)$ classes of Hietarinta's classification of constant 4 by 4 Yang-Baxter solutions [Hietarinta '92].

Tetrahedron Solutions

- Choose $A = X$ and $B = Z$.

$$R(\mu_1,\mu_2,\mu_3) = \mu_1\ X\otimes X\otimes Z + \mu_2\ X\otimes Z\otimes X + \mu_3\ Z\otimes X\otimes X$$

$$= \begin{pmatrix} \cdot & \cdot & \cdot & \mu_3 & \cdot & \mu_2 & \mu_1 & \cdot \\ \cdot & \cdot & \mu_3 & \cdot & \mu_2 & \cdot & \cdot & -\mu_1 \\ \cdot & \mu_3 & \cdot & \cdot & \mu_1 & \cdot & \cdot & -\mu_2 \\ \mu_3 & \cdot & \cdot & \cdot & \cdot & -\mu_1 & -\mu_2 & \cdot \\ \cdot & \mu_2 & \mu_1 & \cdot & \cdot & \cdot & \cdot & -\mu_3 \\ \mu_2 & \cdot & \cdot & -\mu_1 & \cdot & \cdot & -\mu_3 & \cdot \\ \mu_1 & \cdot & \cdot & -\mu_2 & \cdot & -\mu_3 & \cdot & \cdot \\ \cdot & -\mu_1 & -\mu_2 & \cdot & -\mu_3 & \cdot & \cdot & \cdot \end{pmatrix}.$$

- Choose $A = X\left(\frac{\mathbb{1}+Z}{2}\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $B = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$R(\mu_0, \mu_1, \mu_2, \mu_3) = \mu_0\ Z \otimes Z \otimes Z + \mu_1\ A \otimes A \otimes Z$$
$$+\mu_2\ A \otimes Z \otimes A + \mu_3\ Z \otimes A \otimes A$$
$$= \begin{pmatrix} \mu_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\mu_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -\mu_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_3 & \cdot & \cdot & \mu_0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -\mu_0 & \cdot & \cdot & \cdot \\ \mu_2 & \cdot & \cdot & \cdot & \cdot & \mu_0 & \cdot & \cdot \\ \mu_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \mu_0 & \cdot \\ \cdot & -\mu_1 & -\mu_2 & \cdot & -\mu_3 & \cdot & \cdot & -\mu_0 \end{pmatrix}$$

End of Clifford approach.

Except of 'Clifford' we have other sets of solutions:

2) SUSY

3) Majorana

Solutions using SUSY algebras

• **Nilpotent** operators can be realized using $\mathcal{N} = 2$ SUSY algebras

$$q^2 = \left(q^\dagger\right)^2 = 0 \ ; \ \ \{q, q^\dagger\} = h; qq^\dagger = b \ \ ; \ \ q^\dagger q = f.$$

• The two dimensional representation of the supercharges q generates $\mathcal{M}at(2, \mathbb{C})$:

$$q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ ; \ q^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ; \ b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

• Example : SUSY expression for Permutation operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$P = q \otimes q^\dagger + q^\dagger \otimes q + b \otimes b + f \otimes f$$

After Baxterization turns into Yang solution.

• Higher dimensional representations obtained from higher dimensional representations of q.

Baxterizarion and Hamiltonians

Baxterization was developed by:
P. P. Kulish, N. Yu. Reshetikhin E. K. Sklyanin: 1981
L. D. Faddeev, N. Yu. Reshetikhin L. A. Takhtajan: 1987
N. Yu. Reshetikhin: 1987, 1990,1992

In our case:

• Baxterized versions lead to regular R-matrices with additive spectral parameters.

• Baxterizing non-invertible constant 4×4 solutions lead to non-hermitian Hamiltonians:
`https://arxiv.org/pdf/2503.08109`
JHEP05(2025)206

• Baxterizing invertible constant 4×4 solutions lead to both hermitian and non-hermitian Hamiltonians:
`https://arxiv.org/pdf/2508.04315`

Solutions using Majorana fermions

- The Majorana fermion algebra mimics Clifford algebras:

$$\{\gamma_j, \gamma_k\} = 2\delta_{jk}.$$

- A Majorana tetrahedron solution:

$$R_{jkm} = \mathbb{1} + \gamma_j\gamma_k\gamma_m,$$

satisfies the vertex form of the tetrahedron equation.

- Further analysis and other Majorana solutions of all higher simplex equations: https://arxiv.org/pdf/2410.20328 Nuclear Physics B (2025), 0550-3213, 116865.

Ising model can be embedded in our Majorana approach

• The R-matrix: $\mathrm{R}_{jk}(\lambda) = \gamma_j - e^{-2\mathrm{i}\lambda}\,\gamma_k$
The Hamiltonian $\mathrm{H} = \mathrm{i}\sum_{j=1}^{2N-1}\gamma_j\gamma_{j+1} - \mathrm{i}\gamma_{2N}\gamma_1$

B. M. McCoy and T. T. Wu, Harvard University Press 1973;
J. H. Perk and T. T. Wu. Phys. Rev. Lett. 1981

• The transfer matrix contains the Kramers-Wannier duality operator. `https://arxiv.org/pdf/2506.03668`
P. Fendley *et. al* (2016, 2020)

• Fermionic R-matrices can also lead to the 1D **Hubbard model**.
F. Essler *et. al.* (2005)

Solutions of Yang-Baxter equations can be used as gates in quantum circuits

1) J. Phys. A: Math. Theor. 57 445303 (2024)
Adv. Quantum Technol. 2024, 2300345

2) https://arxiv.org/pdf/2406.08320

3) https://arxiv.org/pdf/2307.16781
https://arxiv.org/pdf/2405.16477

Integrable quantum computers.

Summary

• Clifford algebras provide a method to solve the *n*-simplex equations. arXiv:2404.11501v2 [hep-th]

• This framework introduces the anti-*n*-simplex equations and provides its solutions.

• The reflection equations can also be solved with these methods.

• Other solutions using SUSY algebras and Majorana fermions.

• Primary role of my coauthor Pramod Padmanabhan from Indian Institute of Technology.

Thank you for your attention.