Yang-Baxter integrability of bosons hopping on the square lattice with global-range interaction

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Motivation

LETTER

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Quantum phases from competing short- and long-range interactions in an optical lattice

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Insights into complex phenomena in quantum matter can be gained from simulation experiments with ultracold atoms, especially in cases where theoretical characterization is challenging. However, these experiments are mostly limited to short-range collisional interactions; recently observed perturbative effects of long-range interactions were too weak to reach new quantum phases[1,2]. Here we experimentally realize a bosonic lattice model with competing short- and long-range interactions, and observe the appearance of four distinct quantum phases—a superfluid, a supersolid, a Mott insulator and a charge density wave. Our system is based on an atomic quantum gas trapped in an optical lattice inside a high-finesse optical cavity. The strength of the short-range on-site interactions is controlled by means of the optical lattice depth. The long (infinite)-range interaction potential is mediated by a vacuum mode of the cavity[3,4] and is independently controlled by tuning the cavity resonance. When probing the phase transition between the Mott insulator and the charge density wave in real time, we observed a behaviour characteristic of a first-order phase transition. Our measurements have accessed a regime for quantum simulation of many-body systems where the physics is determined by the intricate competition between two different types of interactions and the zero point motion of the particles.

Experiments with cold atoms have contributed in many ways to a stack of about 60 weakly coupled two-dimensional (2D) layers. These 2D layers are then exposed to a square lattice in the x–z plane formed by one free space lattice and one intracavity optical standing wave, both at a wavelength of $\lambda = 785.3$ nm. They create periodic optical potentials of equal depths V_{2D} along both directions, which we will specify in units of the recoil energy $E_R = h^2/2m\lambda^2$, where m denotes the mass of ^{87}Rb. In addition to the lattice potential, the atoms are exposed to an overall harmonic confinement, which results in a maximum density of 2.8 atoms per lattice site at the centre of the trap. The standing wave along the z axis fulfils a second role as it controls long-range interactions via off-resonant scattering into the optical resonator mode. The photons are scattered off the trapped atoms and are delocalized within the cavity mode, thereby mediating atom–atom interactions of infinite range (see Methods). These infinite-range interactions create λ-periodic atomic density–density correlations on the underlying $\lambda/2$-periodic square lattice[4]. The correlations can lead to the breaking of a $\mathbb{Z}_2$-symmetry between the two chequerboard sublattices[23], defined by either even or odd sites, resulting in the appearance of a self-consistent optical potential with alternating strength.

In a wide range of the parameter space, the system can be described by a lattice model with long-range interactions (see Methods and Extended Data Fig. 1), given by:

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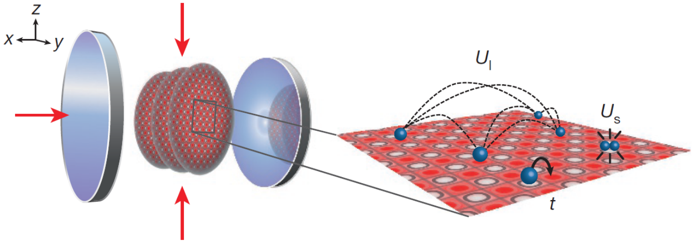

Figure 1 | Illustration of the experimental scheme that realizes a lattice model with on-site and infinite-range interactions. Left, a stack of 2D systems along the y axis is loaded into a 2D optical lattice (red arrows) between two mirrors (shown grey). The cavity induces atom–atom interactions of infinite range. Right, illustration of the competing energy scales: tunnelling t, on-site interactions U_s and long-range interactions U_l.

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ttice model with long-range interactions (see Me
ed Data Fig. 1), given by:

$$\hat{H} = -t \sum_{\langle e,o \rangle} (\hat{b}_e^\dagger \hat{b}_o + \text{h.c.}) + \frac{U_s}{2} \sum_{i \in e,o} \hat{n}_i(\hat{n}_i - 1)$$

$$- \frac{U_l}{K} \left(\sum_e \hat{n}_e - \sum_o \hat{n}_o \right)^2 - \sum_{i \in e,o} \mu_i \hat{n}_i$$

and *o* denote all even and odd lattice sites respectivel

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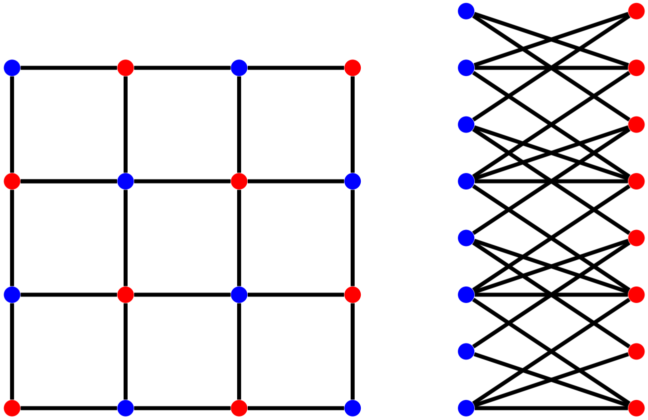
Quotation

There are 'down-to-earth' physicists and chemists who reject lattice models as being unrealistic. In its most extreme form, their argument is that if a model can be solved exactly, then it must be pathological. I think this is defeatist nonsense: ....

Rodney James Baxter,
Exactly Solved Models in Statistical Mechanics,
Academic Press, London, 1982.

Open boundary conditions



Adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & | & \mathcal{B} \\ - & & - \\ \mathcal{B} & | & 0 \end{pmatrix} \cong \begin{pmatrix} \mathcal{B} & | & 0 \\ - & & - \\ 0 & | & -\mathcal{B} \end{pmatrix} .$$

Free-boson Hamiltonian

Let $\{a_j,\, a_j^\dagger : j = 1, \dots, m\} \cup \{b_j,\, b_j^\dagger : j = 1, \dots, m\}$ denote mutually commuting sets of canonical boson operators satisfying

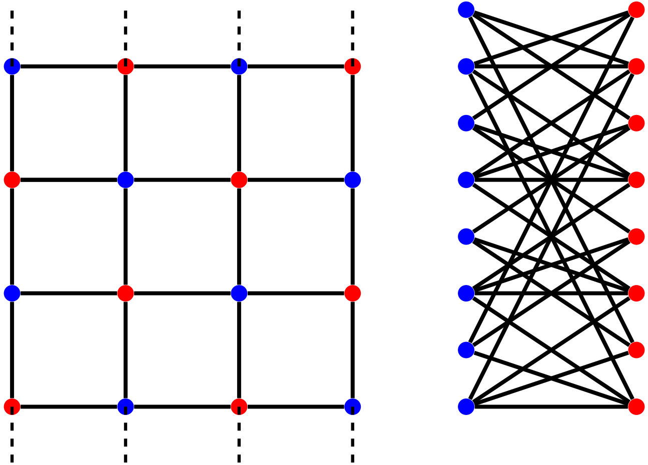
$$[a_j,\, a_k^\dagger] = [b_j,\, b_k^\dagger] = \delta_{jk} I,$$
$$[a_j,\, a_k] = [a_j^\dagger,\, a_k^\dagger] = [b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0.$$

The free-boson Hamiltonian reads $H = \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)$, admitting a set of mutually-commuting conserved operators

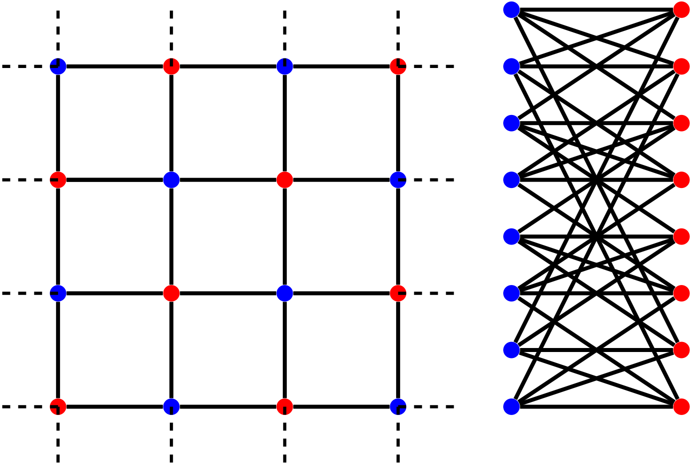
$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

$$C(2p+1) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p+1}(a_j^\dagger b_k + b_j^\dagger a_k)$$

Cylindrical boundary conditions



Toroidal boundary conditions



Free-boson Hamiltonian spectrum

The eigenvalues of the adjacency matrix for open boundary conditions are of the form

$$2\cos\left(\frac{\pi j}{L+1}\right)+2\cos\left(\frac{\pi k}{L+1}\right) \qquad j,k\in\{1,\dots,L\}.$$

The eigenvalues of the adjacency matrix for cylindrical boundary conditions are of the form

$$2\cos\left(\frac{2\pi j}{L}\right)+2\cos\left(\frac{\pi k}{L+1}\right) \qquad j,k\in\{1,\dots,L\}.$$

The eigenvalues of the adjacency matrix for toroidal boundary conditions are of the form

$$2\cos\left(\frac{2\pi j}{L}\right)+2\cos\left(\frac{2\pi k}{L}\right) \qquad j,k\in\{1,\dots,L\}.$$

These provide the single quasi-particle energies.

The interacting Hamiltonian

Let $\{a_j,\, a_j^\dagger : j = 1, \ldots, m\} \cup \{b_j,\, b_j^\dagger : j = 1, \ldots, m\}$ denote mutually commuting sets of canonical boson operators satisfying

$$[a_j,\, a_k^\dagger] = [b_j,\, b_k^\dagger] = \delta_{jk} I,$$
$$[a_j,\, a_k] = [a_j^\dagger,\, a_k^\dagger] = [b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0.$$

For adjacency matrix $\mathcal{A} = \left(\begin{array}{ccc} 0 & | & \mathcal{B} \\ - & & - \\ \mathcal{B} & | & 0 \end{array} \right)$ the Hamiltonian reads

$$H = U(N_a - N_b)^2 + \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)$$

where $N_a = \sum_{j=1}^{m} a_j^\dagger a_j$, $N_b = \sum_{j=1}^{m} b_j^\dagger b_j$. The Hamiltonian admits a set of mutually-commuting conserved operators.

Conserved operators

Explicitly, $[C(y),\ C(z)] = 0$ where

$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

$$C(2p+1) = U\sum_{i=0}^{2p} D(2p,i) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p+1}(a_j^\dagger b_k + b_j^\dagger a_k)$$

with

$$D(2p,i) = \begin{cases} \displaystyle\sum_{j,k,r,q=1}^{m} \mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i}(a_j^\dagger a_q a_r^\dagger a_k + b_j^\dagger b_q b_r^\dagger b_k), & i \text{ even}, \\ \displaystyle\sum_{j,k,r,q=1}^{m} (\mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i} + \mathcal{B}_{jk}^{2p-i}\mathcal{B}_{rq}^{i})a_j^\dagger a_q b_r^\dagger b_k, & i \text{ odd}. \end{cases}$$

Classical Yang-Baxter equation and classical integrability

For $r(u,v) = \sum_{j,k,p,q=1}^{n} r(u,v)_{kq}^{jp} e_j^k \otimes e_p^q$ the classical YBE reads

$$[r_{12}(u,v),\, r_{23}(v,w)] - [r_{21}(v,u),\, r_{13}(u,w)] + [r_{13}(u,w),\, r_{23}(v,w)] = 0.$$

We define the associated *Poisson algebra*

$$\{\mathcal{T}_k^j(u),\, \mathcal{T}_q^p(v)\} = \sum_{\mu=1}^{n} \left(r_{k\mu}^{jp}(u,v)\mathcal{T}_q^{\mu}(v) - r_{kq}^{j\mu}(u,v)\mathcal{T}_\mu^p(v) \right) - \sum_{\mu=1}^{n} \left(r_{q\mu}^{pj}(v,u)\mathcal{T}_k^{\mu}(u) - r_{qk}^{p\mu}(v,u)\mathcal{T}_\mu^j(u) \right).$$

If $B(u)$ satisfies $[B_2(v),\, r_{12}(u,v)] = [B_1(u),\, r_{21}(v,u)]$ we may realise this Poisson algebra through the dual $gl(n)^*$ of $gl(n)$, with Poisson brackets $\{\mathcal{E}_k^j,\, \mathcal{E}_q^p\} = \delta_k^p \mathcal{E}_q^j - \delta_q^j \mathcal{E}_k^p$.

Classical Yang-Baxter equation and classical integrability

The homomorphism is

$$\mathcal{T}^j_k(u) \mapsto B^j_k(u)I + \sum_{p,q=1}^{n} r^{jp}_{kq}(u, v_m)\mathcal{E}^q_p.$$

Set

$$(\mathcal{T}^{(2)})^j_k(u) = \sum_{l=1}^{n} \mathcal{T}^j_l(u)\mathcal{T}^l_k(u),$$

$$(\mathcal{T}^{(r+1)})^j_k(u) = \sum_{l=1}^{n} (\mathcal{T}^{(r)})^j_l(u)\mathcal{T}^l_k(u),$$

$$\mathfrak{t}^{(r)}(u) = \sum_{j=1}^{n} (\mathcal{T}^{(r)})^j_j(u) \quad \Rightarrow \quad \{\mathfrak{t}^{(r)}(u), \mathfrak{t}^{(s)}(v)\} = 0.$$

Expanding $\mathfrak{t}^{(s)}(u) = \sum_j \mathfrak{t}^{(s)}_j u^j$ leads to "Poisson-commuting" functions $\{\mathfrak{t}^{(r)}_j, \mathfrak{t}^{(s)}_k\} = 0.$

Quantisation

The problem to "quantise" Poisson invariants to form a commutative subalgebra of $U(gl(n))$ is well-studied. Towards this goal, note the Lie algebra $gl(n)$ is canonically embedded in $P(gl(n)^*)$ as $P_1(gl(n)^*)$, in that the mapping $\mathcal{E}^j_k \mapsto E^j_k$ between basis elements provides a Lie algebra isomorphism. For $\mathcal{X}_1, \ldots, \mathcal{X}_k \in gl(n)^*$ let the corresponding images under this isomorphism be denoted $X_1, \ldots, Y_k \in gl(n)$. Let S_k denote the symmetric group on k objects. Define the vector space isomorphism $\iota : P_k(gl(n)^*) \to U(gl(n))$ via the following action on products of elements in $gl(n)^*$

$$\iota(1) = I, \qquad \iota(\mathcal{X}_1 \ldots \mathcal{X}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \ldots X_{\sigma(k)},$$

and extended linearly to all of $P(gl(n)^*)$. Set $U_k(gl(n)) = \iota(P_k(gl(n)^*))$. It follows $U(gl(n)) = \bigoplus_{k=0}^{\infty} U_k(gl(n)$.

Let P denote the permutation operator such that

$$P(\mathbf{x} \otimes \mathbf{y}) = \mathbf{y} \otimes \mathbf{x}, \qquad \mathbf{x},\, \mathbf{y} \in \mathbb{C}^n.$$

Set $r(u,v) = \left(\frac{1}{u-v} I \otimes I + \frac{1}{u+v} A \otimes A \right) P$. It may be checked that the classical YBE

$$[r_{12}(u,v),\, r_{23}(v,w)] - [r_{21}(v,u),\, r_{13}(u,w)] + [r_{13}(u,w),\, r_{23}(v,w)] = 0$$

holds provided $A^2 = I$. Moreover, setting $B(u) = uB$ then

$$[B_2(v),\, r_{12}(u,v)] = [B_1(u),\, r_{21}(v,u)]$$

holds provided $AB = -BA$. These conditions are satisfied by choosing $n = 2m$ and $A = \sigma^z \otimes I$, $B = \sigma^x \otimes \mathcal{B}$ for *arbitrary* $\mathcal{B} \in \mathrm{End}(\mathbb{C}^m)$. This solution leads to the conserved operators for the integrable system described earlier.

- Using $r(u, v)$ from the earlier slide, construct the realisation of the Poisson algebra (with $2v = U^{-1}$):

$$\mathcal{T}^j_k(u) \mapsto B^j_k(u)I + \sum_{p,q=1}^{n} r^{jp}_{kq}(u, v)\mathcal{E}^q_p.$$

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- From higher-order "transfer matrix" analogues $\mathfrak{t}^{(s)}(u)$, take *linear and quadratic* Poisson-commuting elements for $s = 2, \ldots, 2m+1$

$$\mathfrak{t}^{(s)}_{s-1}, \quad s \text{ odd}, \qquad\qquad \mathfrak{t}^{(s)}_{s-2} \quad s \text{ even}.$$

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- Generalising the results of Vinberg (1991) shows that these operators quantise to commuting elements of $U(gl(2m))$.

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- Generalising the results of Vinberg (1991) shows that these operators quantise to commuting elements of $U(gl(2m))$.
- Map the elements E^j_k of $gl(2m)$ to operators on Fock space through the Jordan-Schwinger map

$$E^j_k \mapsto a^\dagger_j a_k, \quad j,k \text{ odd}, \qquad E^j_k \mapsto a^\dagger_j b_k, \quad j \text{ odd}, k \text{ even},$$
$$E^j_k \mapsto b^\dagger_j b_k, \quad j,k \text{ even}, \qquad E^j_k \mapsto b^\dagger_j a_k, \quad j \text{ even}, k \text{ odd}.$$

Conserved operators

Explicitly, $[C(y),\ C(z)] = 0$ where

$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

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Canonical transformation

Let X denote a unitary operator that diagonalises $\mathcal{B}$, viz.

$$\sum_{p=1}^{m} X^{\dagger}_{jp} X_{pk} = \delta_{jk}, \qquad \sum_{p,q=1}^{m} X^{\dagger}_{jp} \mathcal{B}_{pq} X_{qk} = \mathcal{E}_j \delta_{jk},$$

with $\{\mathcal{E}_j : j = 1, ..., m\}$ the spectrum of $\mathcal{B}$. Introducing

$$a_k = \sum_{j=1}^{m} X_{kj} c_j, \quad b_k = \sum_{j=1}^{m} X_{kj} d_j, \quad a^{\dagger}_k = \sum_{j=1}^{m} X^{\dagger}_{jk} c^{\dagger}_j, \quad b^{\dagger}_k = \sum_{j=1}^{m} X^{\dagger}_{jk} d^{\dagger}_j,$$

leads to $N_a = \sum_{j=1}^{m} c^{\dagger}_j c_j = N_c$, $N_b = \sum_{j=1}^{m} d^{\dagger}_j d_j = N_d$ and

$$H = U(N_c - N_d)^2 + \sum_{j=1}^{m} \mathcal{E}_j (c^{\dagger}_j d_j + d^{\dagger}_j c_j).$$

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leads to $N_a = \sum_{j=1}^{m} c^{\dagger}_j c_j = N_c$, $N_b = \sum_{j=1}^{m} d^{\dagger}_j d_j = N_d$ and
$H = U(N_c - N_d)^2 + \sum_{j=1}^{m} \mathcal{E}_j (c^{\dagger}_j d_j + d^{\dagger}_j c_j)$. Note that
$\hat{N}_j = c^{\dagger}_j c_j + d^{\dagger}_j d_j$ are conserved operators; let N_j denote their eigenvalues. Then $\sum_{j=1}^{m} N_j = N$ is the total number of particles.

Bethe Ansatz results

For $\{N_1, \ldots, N_m : N_j \in \mathbb{Z}_{\geq 0}\}$ set $|N_1, \ldots, N_m\rangle = (d_1^\dagger)^{N_1} \ldots (d_m^\dagger)^{N_m}|0\rangle$ where $|0\rangle$ denotes the vacuum. The energy eigenvalues are

$E = UN^2 + 4U \sum_{j=1}^{m} \sum_{n=1}^{N} \frac{N_j \mathcal{E}_j^2}{v_n - \mathcal{E}_j^2}$ subject to

$$\sum_{m \neq n}^{N} \frac{2v_n}{v_n - v_m} + \frac{\prod_{j=1}^{m} (v_n - \mathcal{E}_j^2)^{N_j}}{16U^2 \prod_{m \neq n}^{N} (v_n - v_m)} = N - 1 + \sum_{j=1}^{m} \frac{N_j \mathcal{E}_j^2}{v_n - \mathcal{E}_j^2}.$$

for $n = 1, \ldots, N$. The Bethe eigenstates read

$$|v_1, \ldots, v_N; N_1, \ldots, N_m\rangle = \prod_{n=1}^{N} C(v_n)|N_1, \ldots, N_m\rangle,$$

$$C(u) = \frac{1}{2U} I + \sum_{j=1}^{m} \frac{2\varepsilon_j}{u - \varepsilon_j^2} c_j^\dagger d_j.$$

Summary

- Motivated by an optical lattice set-up in a cavity, a model was introduced for bosons on the square lattice with global-range interaction.
- The Hamiltonian, conserved operators, Bethe Ansatz solution follow from the formulation of system through a solution of the classical Yang-Baxter equation.
- Yang-Baxter integrability holds for open, cylindrical, and toroidal boundary conditions.
- The system generalises to models on general bipartite graphs, e.g. hexagonal (a.k.a honeycomb) lattice.

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